August 27 Math 2306 sec. 52 Fall 2021

Section 3: Separation of Variables

Recall that a first order ODE is called **separable** if it has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Such an equation is solved by **separating the variables.**

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

We can also look for possible constant solutions to the separable ODE by trying to solve the equation

$$h(y) = 0.$$

Caveat regarding division by $h(y)$.

The IVP $\quad \dfrac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$

has two solutions

$$y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.$$

The solution $y = x^4/16$ can be obtained by separation of variables.
The constant solution $y(x) = 0$ **cannot**!

Example

In Class Exercise: Take a few minutes and solve the ODE

$$\frac{dy}{dx} = x\sqrt{y}$$

by separating the variables.

If you finish early, try applying the initial condition $y(0)=0$.

$$2y^{1/2} = \frac{x^2}{2} + C$$

one parameter family given implicitly.

soln to IVP $y = \frac{x^4}{16}$

explicit

The family explicitly is

$$y = \left(\frac{x^2}{4} + k\right)^2 \qquad k = c/2$$

$y(x) = 0$ isn't in this family.

Solutions Defined by Integrals

The Fundamental Theorem of Calculus tells us that: If g and $\frac{dy}{dx}$ are continuous on an interval $[x_0, b)$ and x is in this interval, then

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

is a solution of the initial value problem

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

Example

Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

Let's identify the parts.

$g(x) = \sin(x^2)$, $x_0 = \sqrt{\pi}$, $y_0 = 1$

The solution to the IVP is

$$\boxed{y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt}$$

Let's verify this solves the IVP.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

The IC: $y(\sqrt{\pi}) = 1 + \int_{\sqrt{\pi}}^{\sqrt{\pi}} \sin(t^2)\,dt = 1 + 0 = 1$

so $y(\sqrt{\pi}) = 1$.

The ODE: $\frac{dy}{dx} = \frac{d}{dx}\left(1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt\right)$

$$= \frac{d}{dx} 1 + \frac{d}{dx}\int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt$$

$$= 0 + \sin(x^2)$$

$$\frac{dy}{dx} = \sin(x^2)$$

the function satisfies the ODE and the IVP.

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$P = \frac{a_0}{a_1}$$

$$\frac{dy}{dx} + P(x)y = f(x). \qquad f = \frac{g}{a_1}$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the equation
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy=e^x$$

It's not in standard form, but we'll leave it like that for this example.

The left side is the derivative of the product x^2y.

$$\frac{d}{dx}\left[x^2y\right] = x^2\frac{dy}{dx}+2xy$$

So our ODE is

$$\frac{d}{dx}\left[x^2y\right] = e^x$$

The goal is to find y. Integrate

$$\int \frac{d}{dx}\left[x^2 y\right] dx = \int e^x \, dx$$

$$x^2 y = e^x + C$$

$$\boxed{\text{So} \quad y = \frac{e^x + C}{x^2}}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We'll multiply by a function, $\mu(x)$, such that the left side becomes the derivative of the product, μy. I'll assume $\mu(x) > 0$.

$$\mu \frac{dy}{dx} + \mu P(x) y = \mu f(x)$$

We want the left side to be

$$\frac{d}{dx}[\mu y] = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$$

We want

$$\mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + \mu P(x) y$$

This requires $\frac{d\mu}{dx} y = \mu P(x) y$

We get a separable ODE for μ

$$\frac{d\mu}{dx} = \mu P(x).$$

We solve it

$$\int \frac{1}{\mu} d\mu = \int P(x)\, dx$$

$$\ln \mu = \int p(x)\,dx$$

So

$$\mu = e^{\int p(x)\,dx}$$

e to an antiderivative of p

This is called an integrating factor.

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx}f(x)\,dx + C\right)$$