August 27 Math 3260 sec. 51 Fall 2025

1.2 The Vector Space R^3

Vectors in R^3

A **vector** in R^3 is an ordered triple of real numbers,

$$\vec{x} = \langle x_1, x_2, x_3 \rangle,$$

that describe a length, called a *magnitude*, and a direction. The real numbers, x_1, x_2, and x_3 are called the **entries** or **components** of the vector.

When working in R^3, we will consider the set of scalars R.

Standard Representation

As in R^2, we can consider the **standard representation** of a vector in R^3 as a directed line segment. If $\vec{x} = \langle x_1, x_2, x_3 \rangle$, then the standard representation would start at the origin $(0, 0, 0)$ and end at the point (x_1, x_2, x_3).

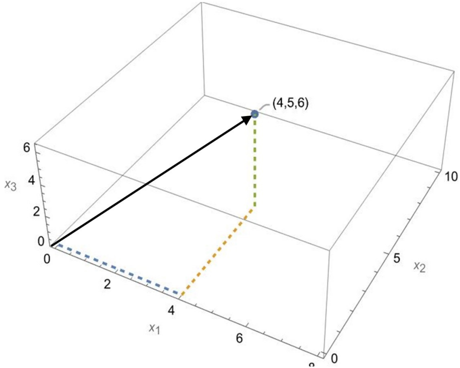

Figure: The standard representation of the vector $\langle 4, 5, 6 \rangle$ is the directed line segment from the point $(0, 0, 0)$ to the point $(4, 5, 6)$ in the Cartesian coordinate system.

Vector Addition & Scalar Multiplication

Let $\vec{x} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$ be vectors in R^3 and let c be a scalar. Then

$$\vec{x} + \vec{y} = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3 \rangle, \quad \text{and}$$

$$c\vec{x} = \langle cx_1, cx_2, cx_3 \rangle.$$

The zero vector $\vec{0}_3 = \langle 0, 0, 0 \rangle$ is the additive identity in the sense that for any vector $\vec{x}$ in R^3

$$\vec{x} + \vec{0}_3 = \vec{0}_3 + \vec{x} = \vec{x}.$$

The additive inverse of the vector $\vec{x}$ is the vector $-\vec{x} = \langle -x_1, -x_2, -x_3 \rangle$, and

$$\vec{x} + (-\vec{x}) = -\vec{x} + \vec{x} = \vec{0}_3.$$

Remark: When we use these operations on vectors in R^3, we refer to the results as **linear combinations**.

Example

Let $\vec{u} = \langle 1, 0, 1 \rangle$ and $\vec{v} = \langle 0, 2, 1 \rangle$. Show that $\vec{x} = \langle 2, -6, -1 \rangle$ is a linear combination of $\vec{u}$ and $\vec{v}$ and identify the weights.

$\vec{x}$ is a linear combination of $\vec{u}$ and $\vec{v}$ if

$\vec{x} = c_1 \vec{u} + c_2 \vec{v}$ for some scalars c_1 and c_2.

$$\langle 2, -6, -1 \rangle = c_1 \langle 1, 0, 1 \rangle + c_2 \langle 0, 2, 1 \rangle$$

$$= \langle c_1, 0, c_1 \rangle + \langle 0, 2c_2, c_2 \rangle$$

$$\langle 2, -6, -1 \rangle = \langle c_1, 2c_2, c_1 + c_2 \rangle$$

This requires $2 = c_1$, $-6 = 2c_2$ and

$-1 = c_1 + c_2$. So, $c_1 = 2$, $c_2 = -3$,

$c_1 + c_2 = 2 + (-3) = -1$.

$\vec{x}$ is a linear combo of $\vec{u}$ and $\vec{v}$,

$\vec{x} = 2\vec{u} - 3\vec{v}$.

1.2.1 Magnitude, Dot Product, & Orthogonality

Magnitude

The **magnitude** or **length** of the vector $\vec{x} = \langle x_1, x_2, x_3 \rangle$ in R^3 is denoted and defined by

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

We call a vector $\vec{u}$ in R^3 such that $\|\vec{u}\| = 1$ a **unit vector.**

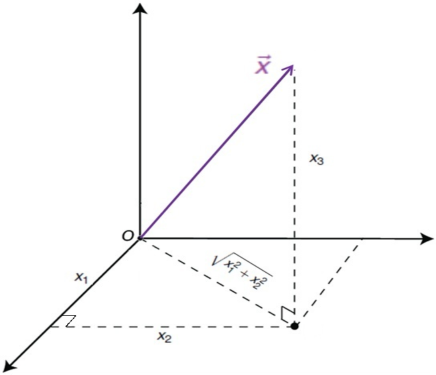

Figure: The magnitude of a vector in R^3 is based on the Pythagorean theorem, just like it is in R^2.

We can use two iterations of the Pythagorean theorem to get

$$\|\vec{x}\|^2 = \left(\sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 = x_1^2 + x_2^2 + x_3^2$$

Dot Product & Orthogonality

The Dot Product

Let $\vec{x} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$ be vectors in R^3. The dot product of these vectors is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Note, as in R^2, the dot product of two vectors is scalar valued.

Orthogonal

The vectors $\vec{x}$ and $\vec{y}$ in R^3 are **orthogonal** if

$$\vec{x} \cdot \vec{y} = 0.$$

Example

Let $\vec{x} = \langle 1, -1, 2 \rangle$. Characterize all vectors $\vec{y} = \langle y_1, y_2, y_3 \rangle$ in R^3 that are orthogonal to $\vec{x}$.
Write the solutions as a linear combination of one or more vectors.

$\vec{y}$ is orthogonal to $\vec{x}$ if $\vec{x} \cdot \vec{y} = 0$.

$\vec{x} \cdot \vec{y} = 1y_1 + (-1)y_2 + 2y_3 = y_1 - y_2 + 2y_3$

$\vec{y}$ is orthogonal to $\vec{x}$ if

$y_1 - y_2 + 2y_3 = 0$. This holds if

$y_1 = y_2 - 2y_3$ for any scalars y_2 and y_3.

Such vectors $\vec{y}$ have the form

$$\vec{y} = \langle y_2 - 2y_3, y_2, y_3 \rangle$$

$$= \langle y_2, y_2, 0 \rangle + \langle -2y_3, 0, y_3 \rangle$$

$$= y_2 \langle 1, 1, 0 \rangle + y_3 \langle -2, 0, 1 \rangle$$

The vectors orthogonal to $\vec{x} = \langle 1, -1, 2 \rangle$ are all linear combos of

$\langle 1, 1, 0 \rangle$ and $\langle -2, 0, 1 \rangle$.

The Geometry of Orthogonality

As in R^2, the pair of vectors $\vec{x}$ and $\vec{y}$ in R^3 satisfy

$$\|\vec{x} + \vec{y}\| = \|\vec{x} - \vec{y}\|$$

if and only if $\vec{x} \cdot \vec{y} = 0$. Orthogonal, nonzero vectors are perpendicular in some plane that is determined by the pair.

(The above claim is left as an exercise!)

1.2.2 Direction

Direction Vector

Let $\vec{x} = \langle x_1, x_2, x_3 \rangle$ be a nonzero vector in R^3. The **direction vector** of $\vec{x}$ is the unit vector

$$\vec{x}_U = \frac{1}{\|\vec{x}\|}\vec{x}.$$

Note that this is defined in the same way as the direction vector in R^2.

The zero vector $\vec{0}_3$ is the only vector that doesn't have a direction.

Direction

We will define the direction cosines in terms of the angles that the standard representation of a nonzero vector makes with respect to the positive x_1, x_2, and x_3-axes. So a nonzero vector in R^3 will have three direction cosines.

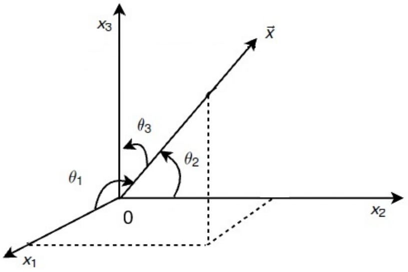

Figure: The standard representation of a vector makes angles with each of the three positive coordinate axes.

Direction Cosines

If $\vec{x} = \langle x_1, x_2, x_3 \rangle$ is a nonzero vector, then the **direction cosines** are

$$\cos\theta_1 = \frac{x_1}{\|\vec{x}\|}, \quad \cos\theta_2 = \frac{x_2}{\|\vec{x}\|}, \quad \text{and} \quad \cos\theta_3 = \frac{x_3}{\|\vec{x}\|}.$$

The **direction angles** are

$$\theta_1 = \cos^{-1}\left(\frac{x_1}{\|\vec{x}\|}\right), \quad \theta_2 = \cos^{-1}\left(\frac{x_2}{\|\vec{x}\|}\right), \quad \text{and} \quad \theta_3 = \cos^{-1}\left(\frac{x_3}{\|\vec{x}\|}\right).$$

As we saw in R^2, in R^3

$$\vec{x}_U = \langle \cos\theta_1, \cos\theta_2, \cos\theta_3 \rangle, \quad \text{and} \quad \vec{x} = \|\vec{x}\|\vec{x}_U.$$

Distance

If $\vec{x}$ and $\vec{y}$ are any two vectors in R^3, then the distance bewteen $\vec{x}$ and $\vec{y}$ is

$$\text{dist}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|.$$

1.3 The Vector Space R^n

We define objects called **vectors** in R^n. A vector is an ordered n-tuple

$$\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle.$$

The real numbers, $x_1, x_2, \ldots, x_n$ are called the **entries** or **components** of $\vec{x}$.

When working in R^n, we will work with **scalars.** For us, scalars will be real numbers, elements of R.

Vector Addition & Scalar Multiplication

Let $\vec{x} = \langle x_1, x_2, \ldots, x_n\rangle$ and $\vec{y} = \langle y_1, y_2, \ldots, y_n\rangle$ be vectors in R^n and let c be a scalar. Then

$$\vec{x} + \vec{y} = \langle x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\rangle, \quad \text{and}$$

$$c\vec{x} = \langle cx_1, cx_2, \ldots, cx_n\rangle.$$

The zero vector $\vec{0}_n = \underbrace{\langle 0, 0, \ldots, 0\rangle}_{n \text{ zeros}}$ is the additive identity in the sense that for any vector $\vec{x}$ in R^n

$$\vec{x} + \vec{0}_n = \vec{0}_n + \vec{x} = \vec{x}.$$

The additive inverse of the vector $\vec{x}$ is the vector $-\vec{x} = \langle -x_1, -x_2, \ldots, -x_n\rangle$, and

$$\vec{x} + (-\vec{x}) = -\vec{x} + \vec{x} = \vec{0}_n.$$

Remark: When we form vectors using these operations, we refer to the result as a **linear combination**.

Magnitude

Given a vector $\vec{x} = \langle x_1, x_2, \dots, x_n\rangle$ in R^n, the **magnitude** of x is the nonnegative number

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

A vector $\vec{u}$ such that $\|\vec{u}\| = 1$ is called a **unit vector**.

Dot Product & Orthogonality

The **dot product** of the pair of vectors $\vec{x} = \langle x_1, x_2, \dots, x_n\rangle$ and $\vec{y} = \langle y_1, y_2, \dots, y_n\rangle$ in R^n is the scalar

$$\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

We say that two vectors $\vec{x}$ and $\vec{y}$ in R^n are **orthogonal** if

$$\vec{x}\cdot\vec{y} = 0.$$

Example

Let $\vec{x} = \langle 1, 0, 2, -3, 4 \rangle$, $\vec{y} = \langle 1, 1, -1, 4, 5 \rangle$, and $\vec{z} = \langle 2, 1, 3, 0, p \rangle$.

1. If $\vec{x}$, $\vec{y}$ and $\vec{z}$ are in R^n, what is n?

   $n = 5$

2. Determine $\|\vec{x}\|$.

   $\|\vec{x}\|^2 = 1^2 + 0^2 + 2^2 + (-3)^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

   $\|\vec{x}\| = \sqrt{30}$

Example

Let $\vec{x} = \langle 1, 0, 2, -3, 4 \rangle$, $\vec{y} = \langle 1, 1, -1, 4, 5 \rangle$, and $\vec{z} = \langle 2, 1, 3, 0, p \rangle$.

3. Evaluate $\vec{x} \cdot \vec{y}$.

$\vec{x} \cdot \vec{y} = 1(1) + (0)(1) + 2(-1) + (-3)(4) + 4(5)$

$= 1 - 2 - 12 + 20 = 7$

4. For what value(s) of p, if any, are $\vec{x}$ and $\vec{z}$ orthogonal?

$\vec{x} \cdot \vec{z} = 1(2) + 0(1) + 2(3) + (-3)(0) + 4(p)$

$= 2 + 6 + 4p = 8 + 4p.$

They're orthogonal if $8 + 4p = 0 \Rightarrow p = -2.$

1.3.1 Algebraic Properties of the Dot Product

The dot product is sometimes called a **scalar product** because it acts on two vectors to produce a scalar. It is an example of something called an **inner product** because it satisfies the following algebraic properties:

For every $\vec{x}$, $\vec{y}$ and $\vec{z}$ in R^n and scalar c in R

- $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$ (commutative property)
- $(c\vec{x}) \cdot \vec{y} = \vec{x} \cdot (c\vec{y}) = c(\vec{x} \cdot \vec{y})$ (scalars factor)
- $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$ (distributive property)
- $\vec{x} \cdot \vec{x} \geq 0$ with $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}_n$.

This looks like a previous slide, but there's one small difference. Before, we considered vectors in R^2, and now we're considering vectors in R^n. But the properties are the same!