August 28 Math 2306 sec. 52 Spring 2023

Section 3: Separation of Variables

Separable Differential Equations

Recall that a first order equation of the form

$$\frac{dy}{dx} = g(x)h(y) \tag{1}$$

is called **separable**.

- If $h(c) = 0$, the $y = c$ is a constant solution to (1).
- The equation (1) may be solved by separation of variables,

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

Solutions Expressed as Integrals

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\, dt$$

is a solution of the initial value problem $\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$.

Generalizing

If p and g are sufficiently continuous then

$$\int_{y_0}^{y} p(z)\, dz = \int_{x_0}^{x} g(t)\, dt \quad \text{solves} \quad \frac{dy}{dx} = \frac{g(x)}{p(y)}, \quad y(x_0) = y_0$$

Example: Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

Let's identify the pieces.

$g(x) = \sin(x^2)$, $x_0 = \sqrt{\pi}$, $y_0 = 1$

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

The solution to the IVP is

$$\boxed{y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt}$$

Section 4: First Order Equations: Linear

Recall that a first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Standard Form

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$P(x) = \frac{a_0(x)}{a_1(x)}$

$$\frac{dy}{dx} + P(x)y = f(x).$$

$f(x) = \frac{g(x)}{a_1(x)}$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

The Solutions of $\frac{dy}{dx} + P(x)y = f(x)$

The solution to a first order linear ODE always has the same basic structure

$$y(x) = y_c(x) + y_p(x) \quad \text{where}$$

- $\boxed{y_c}$ is called the **complementary** solution. The complementary solution solves **associated homogeneous** equation, $\frac{dy}{dx} + P(x)y = 0$, and
- $\boxed{y_p}$ is called the **particular** solution. The particular solution depends heavily on f and is zero if $f(x) = 0$.

With higher order equations, we'll have to find y_c and y_p separately, but for first order equations we have a process for finding the whole solution.

Motivating Example

Find the solutions of $\quad x^2\dfrac{dy}{dx}+2xy=e^x.$

This is not in standard form, but we'll work with it as is. The left side is the derivative of a product, it is $\frac{d}{dx}\left(x^2y\right)$.

$$\frac{d}{dx}\left(x^2y\right)=x^2\frac{dy}{dx}+2xy$$

so the ODE is

$$\frac{d}{dx}\left(x^2y\right)=e^x$$

The goal is to find y.

Integrate both sides

$$\int \frac{d}{dx}\left(x^2 y\right) dx = \int e^x \, dx$$

$$x^2 y = e^x + C$$

Assuming $x \neq 0$

$$y = \frac{e^x + C}{x^2}$$

1-parameter family of solutions

$$y = \frac{C}{x^2} + \frac{e^x}{x^2}$$

$$y_c = \frac{C}{x^2} \quad \text{and} \quad y_p = \frac{e^x}{x^2}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

Well manipulate the equation to make the left side collapse as a product rule. We multiply the equation by a function $\mu(x)$ so that the left side becomes $\frac{d}{dx}(\mu y)$. Assume μ exists and $\mu(x) > 0$.

$$\mu\left(\frac{dy}{dx} + P(x)y\right) = \mu f(x)$$

$$\mu \frac{dy}{dx} + P(x)\mu y = \mu f(x)$$

$$\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$$

we need $\quad \mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + P(x)\mu y$

This requires the purple terms to match.

$$\frac{d\mu}{dx} y = P(x)\mu y \qquad \text{cancel } y$$

$$\frac{d\mu}{dx} = P(x)\mu \qquad \text{separable ODE for } \mu$$

Solve for μ

$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\, dx$$

$$\ln \mu = \int P(x)\, dx$$

$$\boxed{\mu = e^{\int P(x)\, dx}}$$

This is called an integrating factor.

Integrating Factor

Integrating Factor

For the first order, linear ODE in standard form

$$\frac{dy}{dx} + P(x)y = f(x),$$

the integrating factor

$$\mu(x) = \exp\left(\int P(x)\,dx\right).$$

Let's list the steps involved in solving a first order linear ODE.