August 28 Math 2306 sec. 53 Fall 2024

Section 4: First Order Equations: Linear

Recall that a first order linear equation is one that has the form[1]

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

In **standard form**, a first order linear equation looks like

$$\frac{dy}{dx} + P(x)y = f(x).$$

We'll assume that P and f are continuous on the domain of the solution. The solution will have the basic structure

$$y(x) = y_c(x) + y_p(x)$$

where y_c is called the **complementary** solution and y_p is called the **particular** solution.

[1] It's called homogeneous if $g(x) = 0$ and nonhomogeneous otherwise.

Solution Process 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)f(x)\,dx$$

$$y(x) = e^{-\int P(x)\,dx} \left(\int e^{\int P(x)\,dx} f(x)\,dx + C \right)$$

Example

Solve the initial value problem

$$x\frac{dy}{dx} - y = 2x^2, \; x > 0 \quad y(1) = 5$$

Divide by x to get standard form.

$$\frac{dy}{dx} - \frac{1}{x} y = 2x \quad , \quad P(x) = \frac{-1}{x}$$

$$\mu = e^{\int P(x)\,dx} = e^{\int \frac{-1}{x} dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln x}$$

$$\mu = e^{\ln x^{-1}} = x^{-1}$$

$$x^{-1}\left(\frac{dy}{dx} - \frac{1}{x} y\right) = x^{-1}(2x)$$

$$\frac{d}{dx}\left(x^{-1}y\right) = 2$$

Integrate with respect to x

$$\int \frac{d}{dx}\left(x^{-1}y\right)dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$\Rightarrow \quad y = \frac{2x+C}{x^{-1}} = x(2x+C)$$

$$y = 2x^2 + Cx$$

This is a one-parameter family of solutions to the ODE.

Apply $y(1) = 5$

$$y(1) = 2(1^2) + C(1) = 5$$

$$2 + C = 5 \implies C = 3$$

The solution to the IVP is

$$y = 2x^2 + 3x.$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

$y = 2x^2 + 3x, \quad y' = 4x + 3$

$xy' - y \stackrel{?}{=} 2x^2.$

$x(4x+3) - (2x^2 + 3x) \stackrel{?}{=} 2x^2$

$4x^2 + \cancel{3x} - 2x^2 - \cancel{3x} \stackrel{?}{=} 2x^2$

$4x^2 - 2x^2 \stackrel{?}{=} 2x^2$

$2x^2 = 2x^2$ ✓

This shows that $y = 2x^2 + 3x$ solves the ODE.

Why don't we need the "$+C$" in μ?

Why was it OK to take

$$\mu = e^{-\ln(x)} = x^{-1} \quad \text{instead of} \quad \mu = e^{-\ln(x)+C} = e^C x^{-1}?$$

Look at what happens to the factor e^C.

$$e^C x^{-1}\left(y' - \frac{1}{x}y\right) = e^C x^{-1}(2x) \quad \Longrightarrow \quad \frac{d}{dx}\left(e^C x^{-1} y\right) = 2e^C.$$

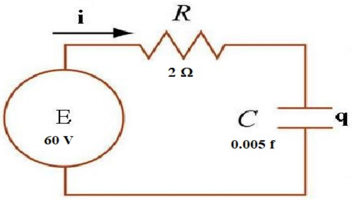
The constant can be factored out of the derivative and cancelled on both sides!

$$e^C \frac{d}{dx}\left(x^{-1}y\right) = 2e^C \quad \Longrightarrow \quad \cancel{e^C}\frac{d}{dx}\left(x^{-1}y\right) = 2\cancel{e^C}$$

Again, we end up with $\dfrac{d}{dx}\left(x^{-1}y\right) = 2$.

When computing the integrating factor, μ, I'll always take the added constant to be zero.

Steady and Transient States



$i = \frac{dq}{dt}$

Figure: The charge $q(t)$ on the capacitor in the given curcuit satisfies a first order linear equation.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0.$$

Solve this IVP for the charge $q(t)$ on the capacitor for $t > 0$.

In standard form $\frac{dq}{dt} + 100q = 30$

$P(t) = 100$

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0$$

$$\mu = e^{\int p(t)\,dt} = e^{\int 100\,dt} = e^{100t}$$

$$e^{100t}\left(\frac{dq}{dt} + 100q\right) = e^{100t}(30)$$

$$\frac{d}{dt}\left(e^{100t} q\right) = 30e^{100t}$$

$$\int \frac{d}{dt}\left(e^{100t} q\right) dt = \int 30 e^{100t}\, dt$$

$$e^{100t} q = \frac{30}{100} e^{100t} + k$$

$$q = \frac{\frac{3}{10} e^{100t} + k}{e^{100t}} = \frac{3}{10} \frac{e^{100t}}{e^{100t}} + \frac{k}{e^{100t}}$$

$$q = \frac{3}{10} + k e^{-100t}$$

a one parameter family of solutions to the ODE. Apply $q(0) = 0$

$$q(0) = \frac{3}{10} + k e^{0} = 0$$

$$\frac{3}{10} + k = 0 \Rightarrow k = \frac{-3}{10}$$

The charge on the capacitor is $\boxed{q = \frac{3}{10} - \frac{3}{10} e^{-100t}}$

Steady and Transient States

Note that the solution, the charge, consists of a complementary and a particular solution, $q = q_p + q_c$.

$$q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}$$

$$q_c(t) = -\frac{3}{10}e^{-100t} \quad \text{and} \quad q_p(t) = \frac{3}{10}$$

Evaluate the limit

$$\lim_{t\to\infty} q_c(t) = \lim_{t\to\infty} \frac{-3}{10} e^{-100t} = 0$$

$$q(t) \approx q_p(t)$$

Steady and Transient States

The complementary solution contains the information given by the initial condition, and for some physical systems like this the complementary solution decays.

Definition: Such a decaying complementary solution is called a **transient state**.

Note that due to this decay, after a while $q(t) \approx q_p(t)$.

Definition: Such a corresponding particular solution is called a **steady state**.