August 29 Math 2306 sec. 51 Fall 2022

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$P(x) = \frac{a_0(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)y = f(x).$$

$$f(x) = \frac{g(x)}{a_1(x)}.$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the equation
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy = e^x$$

This is not in standard form, but we'll make an exception. The left side is the derivative of the product, x^2y.

$$\frac{d}{dx}\left(x^2y\right) = x^2\frac{dy}{dx} + 2xy$$

The ODE is

$$\frac{d}{dx}\left(x^2y\right) = e^x$$

The goal is to get y.

Integrate

$$\int \frac{d}{dx}(x^2 y)\, dx = \int e^x \, dx$$

$$x^2 y = e^x + C$$

Divide by x^2

$$y = \frac{e^x}{x^2} + \frac{C}{x^2}$$

This is the general solution

Here, $y_c = \frac{C}{x^2}$ and $y_p = \frac{e^x}{x^2}$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We want the left side to collapse as one derivative. We'll multiply the ODE by a function $\mu(x)$, and choose μ so that the left side becomes $\frac{d}{dx}\left(\mu(x)y\right)$.

We assume $\mu(x)$ exists and that $\mu(x) > 0$.

The ODE is

$$\mu\frac{dy}{dx} + P(x)\mu y = \mu f(x)$$

Note $\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$

Set the left side of the ODE to this

$$\mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + P(x)\mu y$$

We need

$$\frac{d\mu}{dx} y = P(x)\mu y$$

Cancel the y

$$\frac{d\mu}{dx} = P(x)\mu$$

This is 1^{st} order separable.

$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\int \frac{1}{\mu}\,d\mu = \int P(x)\,dx$$

$$\ln \mu = \int P(x)\,dx$$

$$\mu = e^{\int P(x)\,dx}$$

This is called an integrating factor

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Solve the IVP

$$\frac{dy}{dx} + P(x)\, y = f(x)$$

$$x\frac{dy}{dx} - y = 2x^2, \; x > 0 \quad y(1) = 5$$

Put it in standard form (divide by x)

$$\frac{dy}{dx} - \frac{1}{x}\, y = \frac{2x^2}{x} = 2x$$

$$P(x) = \frac{-1}{x}, \quad \mu = e^{\int P(x)\,dx}$$

$$\int P(x)\,dx = \int -\frac{1}{x}\,dx = -\ln x$$

$$\mu = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

Multiply by μ:

$$x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = x^{-1}(2x)$$

$$\frac{d}{dx}\left(x^{-1}y\right) = 2$$

$$\int \frac{d}{dx}\left(x^{-1}y\right)dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$y = \frac{2x+C}{x^{-1}} = 2x^2 + Cx$$

$y = 2x^2 + Cx$ is a 1-parameter family of solutions to the ODE.

Now apply $y(1) = 5$.

$$y(1) = 2(1)^2 + C(1) = 5$$

$$2 + C = 5 \implies C = 3$$

$$\boxed{\text{The solution to the IVP is } \quad y = 2x^2 + 3x}$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

$y = 2x^2 + 3x\ ,\quad y' = 4x + 3$

$x\frac{dy}{dx} - y \stackrel{?}{=} 2x^2$

$x(4x+3) - (2x^2 + 3x) \stackrel{?}{=} 2x^2$

$4x^2 + \cancel{3x} - 2x^2 - \cancel{3x} \stackrel{?}{=} 2x^2$

$2x^2 = 2x^2$ ✓ we get an identity.