August 29 Math 3260 sec. 53 Fall 2025

1.3.2 Span

We'll recall that for vectors $\vec{x} = \langle x_1, x_2, \dots, x_n \rangle$ and $\vec{y} = \langle y_1, y_2, \dots, y_n \rangle$ in R^n and scalar c in R, vector addition and scalar multiplication are defined by

$$\vec{x} + \vec{y} = \langle x_1 + y_1, x_2 + y_2, \dots, x_n + y_n \rangle, \quad \text{and} \quad c\vec{x} = \langle cx_1, cx_2, \dots, cx_n \rangle.$$

Linear Combination in R^n

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a set of one or more ($k \geq 1$) vectors in R^n. A **linear combination** of these vectors is any vector of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k,$$

where $c_1, \dots, c_k$ are scalars. The coefficients, $c_1, \dots, c_k$, are often called the **weights**. They can also be called **coefficients**.

1.3.2 Span: Let's Recall Some Examples

Last week, we determined that the **set of all linear combinations of the vector $\vec{e}_1 = \langle 1, 0\rangle$ in R^2** is the horizontal (i.e., x_1) axis.

In a previous example, we showed that the set of all vectors in R^3 that are orthogonal to $\vec{x} = \langle 1, -1, 2\rangle$ in R^3 have the form

$$\vec{y} = y_2\langle 1, 1, 0\rangle + y_3\langle -2, 0, 1\rangle,$$

where the scalars y_2 and y_3 can be any real numbers. We can call this **the set of all linear combinations of $\langle 1, 1, 0\rangle$ and $\langle -2, 0, 1\rangle$.**

We'll call a vector like $\langle 1, 1, 0\rangle$ (with numbers but not variables in it) a **fixed vector**.

1.3.2 Span

Definition: Span

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of one or more ($k \geq 1$) vectors in R^n. The set of all possible linear combinations of the vectors in S is called the **subspace of R^n spanned by** S. We often refer to this as **span** of S. It is denoted

$$\text{Span}(S) \quad \text{or by} \quad \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}.$$

Example: If S is a set of one vector, $S = \{\vec{v}\}$, then

$$\text{Span}(S) = \{c\vec{v} \mid c \in R\}.$$

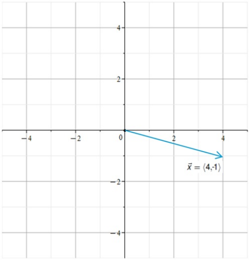
That is, $\text{Span}(S) = \text{Span}\{\vec{v}\}$ is the set of all scalar multiples of the vector $\vec{v}$. If $S = \{\vec{u}, \vec{v}\}$, so S is a set of two vectors, then

$$\text{Span}(S) = \{c_1\vec{u} + c_2\vec{v} \mid c_1, c_2 \in R\},$$

and so forth.

Example

If $S = \{\langle \frac{1}{4}, 1 \rangle\}$, then $\text{Span}(S)$ is the set of all vectors that are orthogonal to $\vec{x} = \langle 4, -1 \rangle$.



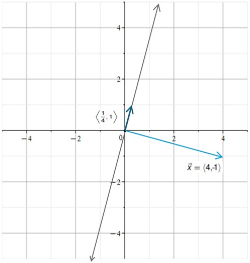


Figure: For nonzero $\vec{v}$, $\text{Span}\{\vec{v}\}$ is a line in R^2 through the origin and parallel to $\vec{v}$. Left: Standard representation of $\langle 4, -1 \rangle$. Right: $\langle 4, -1 \rangle$, $\langle 1/4, 1 \rangle$, and $\text{Span}\{\langle 1/4, 1 \rangle\}$.

Example

If $S = \{\langle 1,1,0\rangle, \langle -2,0,1\rangle\}$, then

Span(S) is the set of all vectors in R^3 that are orthogonal to $\vec{x} = \langle 1,-1,2\rangle$.

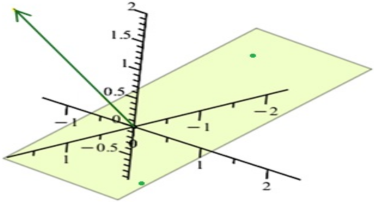


Figure: The set of all vectors orthogonal to $\langle 1,-1,2\rangle$ is a plane containing the origin. This plane contains the points $(1,1,0)$ and $(-2,0,1)$.

What does it mean for a vector to be in a subspace spanned by a set?

If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$, then to say that the vector $\vec{x}$ is in $\mathrm{Span}(S)$—written $\vec{x} \in \mathrm{Span}(S)$, means that there exist some scalars $c_1, \dots, c_k$ such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k.$$

The symbol "$\in$" means "*is an element of*." It can be read as "*is in*," e.g.,

$$x \in \cdots \quad \text{"}x \text{ is in..."}$$

Example: Let $S = \{\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle\}$.

1. Which of the following vectors are in S?

   $\langle 1, 0, 0\rangle$ ✓, $\langle 2, 0, 0\rangle$ ✗, $\langle 0, 1, 0\rangle$ ✓, $\langle 1, 1, 0\rangle$ ✗

2. Which of the following vectors[1] are in $\mathrm{Span}(S)$?

   $\langle 1, 0, 0\rangle$ ✓, $\langle 2, 0, 0\rangle$ ✓, $\langle 0, 1, 0\rangle$ ✓, $\langle 1, 1, 0\rangle$ ✓

3. Are there vectors in R^3 that are not in $\mathrm{Span}(S)$?

$\vec{x} \in \mathrm{Span}(S)$ if $\vec{x} = c_1\langle 1,0,0\rangle + c_2\langle 0,1,0\rangle$ for some $c_1, c_2 \in \mathbb{R}$.
$= \langle c_1, c_2, 0\rangle$

There are vectors in $\mathbb{R}^3$ that are not in Span(S).
$\langle 0,0,1\rangle$ is an example.

[1] Note that $\mathrm{Span}(S) = \{c_1\langle 1, 0, 0\rangle + c_2\langle 0, 1, 0\rangle \mid c_1, c_2 \in R\}$.

Example

Let $S = \{\vec{u}, \vec{v}\}$ where $\vec{u} = \langle 1, 1 \rangle$ and $\vec{v} = \langle 1, -1 \rangle$.

1. Show that the vector $\vec{x} = \langle 2, 3 \rangle$ is in $\mathrm{Span}(S)$.
2. Show that $R^2 = \mathrm{Span}(S)$.

1. $\vec{x} \in \mathrm{Span}(S)$ if $\vec{x} = c_1\vec{u} + c_2\vec{v}$ for some scalars c_1, c_2.

$\langle 2, 3 \rangle = c_1 \langle 1, 1 \rangle + \cdot \ \langle 1, -1 \rangle$

$= \langle c_1 + c_2, c_1 - c_2 \rangle$

This is true if $c_1 + c_2 = 2$ and

$c_1 - c_2 = 3$

add: $2c_1 = 5 \Rightarrow c_1 = \frac{5}{2}$

subtract: $2c_2 = -1 \Rightarrow c_2 = \frac{-1}{2}$

So $\vec{x} \in \text{Span}(S)$ and $\vec{x} = \frac{5}{2}\vec{u} - \frac{1}{2}\vec{v}$.

Check: $\langle 2,3\rangle \stackrel{?}{=} \frac{5}{2}\langle 1,1\rangle - \frac{1}{2}\langle 1,-1\rangle$ ✓

$= \langle \frac{5}{2} - \frac{1}{2}, \frac{5}{2} + \frac{1}{2}\rangle = \langle \frac{4}{2}, \frac{6}{2}\rangle = \langle 2,3\rangle$

2. Show that $R^2 = \text{Span}(S)$.

We need to show that $\vec{x} \in \text{Span}(S)$ for any vector $\vec{x} \in R^2$. Let $\vec{x} = \langle x_1, x_2\rangle$ for any choice of x_1 and x_2. We need c_1, c_2 such that $\vec{x} = c_1\vec{u} + c_2\vec{v}$.

$$\langle x_1, x_2\rangle = c_1\langle 1,1\rangle + c_2\langle 1,-1\rangle$$

$$= \langle c_1 + c_2, c_1 - c_2\rangle$$

This holds if

$$c_1 + c_2 = x_1$$
$$c_1 - c_2 = x_2$$

add $\quad 2c_1 = x_1 + x_2 \Rightarrow c_1 = \dfrac{x_1 + x_2}{2}$

subtract $\quad 2c_2 = x_1 - x_2 \Rightarrow c_2 = \dfrac{x_1 - x_2}{2}$

For any $\vec{x} \in \mathbb{R}^2$

$$\langle x_1, x_2 \rangle = \left(\frac{x_1 + x_2}{2}\right)\langle 1, 1 \rangle + \left(\frac{x_1 - x_2}{2}\right)\langle 1, -1 \rangle.$$

Check: $\left(\frac{x_1+x_2}{2}\right)\langle 1,1\rangle + \left(\frac{x_1-x_2}{2}\right)\langle 1,-1\rangle$

$$= \left\langle \frac{x_1+x_2}{2}, \frac{x_1+x_2}{2} \right\rangle + \left\langle \frac{x_1-x_2}{2}, \frac{-(x_1-x_2)}{2} \right\rangle$$

$$= \left\langle \frac{x_1+x_2}{2} + \frac{x_1-x_2}{2}, \frac{x_1+x_2}{2} - \frac{x_1-x_2}{2} \right\rangle$$

$$= \left\langle \frac{x_1+x_2+x_1-x_2}{2}, \frac{x_1+x_2-x_1+x_2}{2} \right\rangle$$

$$= \left\langle \frac{2x_1}{2}, \frac{2x_2}{2} \right\rangle = \langle x_1, x_2 \rangle \quad \checkmark$$