August 30 Math 2306 sec. 51 Fall 2021

Section 4: First Order Equations: Linear

Recall that we were interested in first order linear equations. Such an equation in **standard form** looks like

$$\frac{dy}{dx} + P(x)y = f(x).$$

We're assuming that P and f are continuous on the domain of definition. The solution, that we'll call the **general solution** will have the form

$$y = y_c + y_p$$

where y_c is called the **complementary** solution and y_p is called the **particular** solution. If $f(x) = 0$, we call the ODE **homogeneous**[1]. Otherwise, we call it **nonhomogeneous**.

[1] $y_p = 0$ if the ODE is homogeneous.

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \; x > 0 \quad y(1) = 5$$

Divide by x to get standard form.

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{2x^2}{x} = 2x$$

$$\frac{dy}{dx} + P(x)y \qquad P(x) = \frac{-1}{x}$$

The integrating factor $\mu = e^{\int P(x)\,dx}$

$$\int P(x)\,dx = \int \frac{-1}{x}\,dx = -\ln x$$

$$\mu = e^{\int P(x)\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

Mult. the ODE by μ

$$x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = x^{-1}(2x)$$

$$\frac{d}{dx}\left[x^{-1}y\right] = 2$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

hence $\quad y = \dfrac{2x+C}{x^{-1}} = x(2x+C)$

The solution to the ODE is

$$y = \underbrace{2x^2}_{y_p} + \underbrace{Cx}_{y_c}$$

Apply the I.C. $\quad y(1) = 5$

$$y(1) = 2(1)^2 + C(1) = 5$$

$$2 + C = 5 \implies C = 3$$

The solution to the IVP

$$y = 2x^2 + 3x.$$

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

$y = 2x^2 + 3x$, $y' = 4x + 3$

$x\frac{dy}{dx} - y \stackrel{?}{=} 2x^2$

$x(4x+3) - (2x^2 + 3x) \stackrel{?}{=} 2x^2$

$4x^2 + \cancel{3x} - 2x^2 - \cancel{3x} \stackrel{?}{=} 2x^2$

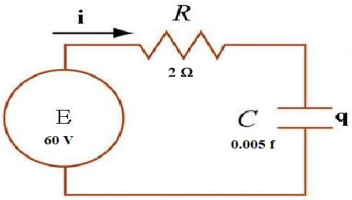
$2x^2 = 2x^2.$

yep! this is an identity.

Steady and Transient States

Current

$i = \frac{dq}{dt}$



q - charge

Figure: The charge $q(t)$ on the capacitor in the given curcuit satisfies a first order linear equation.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0.$$

Standard form

$$\frac{dq}{dt} + 100\, q = 30 \qquad P(t) = 100$$

$$\mu = e^{\int p(t)\,dt} = e^{\int 100\,dt} = e^{100t}$$

$$e^{100t}\left(\frac{dq}{dt} + 100q\right) = e^{100t}(30)$$

$$\frac{d}{dt}\left[e^{100t} q\right] = 30e^{100t}$$

$$\int \frac{d}{dt}\left[e^{100t} q\right] dt = \int 30e^{100t}\,dt$$

$$e^{100t} q = \frac{30}{100} e^{100t} + K$$

$$q = \frac{\frac{3}{10}e^{100t} + k}{e^{100t}} = \frac{3}{10} + ke^{-100t}$$

$$\int e^{at}\,dt = \frac{1}{a}e^{at} + C$$

a - nonzero constant.

Use $q(0) = 0$

$$q(0) = \frac{3}{10} + k e^{0} = 0$$

$$k = \frac{-3}{10}$$

$$\boxed{\begin{gathered}\text{The charge on the capacitor} \\ q(t) = \frac{3}{10} - \frac{3}{10} e^{-100t}\end{gathered}}$$

Steady and Transient States

Note that the solution, the charge, consists of a complementary and a particular solution, $q = q_p + q_c$.

$$q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}$$

$$q_c(t) = -\frac{3}{10}e^{-100t} \quad \text{and} \quad q_p(t) = \frac{3}{10}$$

Evaluate the limit

$$\lim_{t\to\infty} q_c(t) = \lim_{t\to\infty} \frac{-3}{10} e^{-100t} = 0$$

Steady and Transient States

The complementary solution contains the information given by the initial condition, and for some physical systems like this the complementary solution decays.

Definition: Such a decaying complementary solution is called a **transient state**.

Note that due to this decay, after a while $q(t) \approx q_p(t)$.

Definition: Such a corresponding particular solution is called a **steady state**.

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Divide by y^n $\qquad y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = f(x)$

Let $u = y^{1-n}$. Then $\frac{du}{dx} = (1-n)\,y^{1-n-1}\frac{dy}{dx}$

so $\frac{du}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}$

Let's multiply the ODE by $1-n$.

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)P(x)\,y^{1-n} = (1-n)f(x)$$

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)f(x)$$

This is 1st order linear of the form

$$\frac{du}{dx} + P_1(x)\,u = f_1(x)$$

where $P_1(x) = (1-n)P(x)$ and

$$f_1(x) = (1-n)f(x).$$

$$u = y^{1-n} \implies y = u^{\frac{1}{1-n}}$$

Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

$$y' - y = -e^{2x}y^3, \quad n = 3$$

so $\quad u = y^{1-3} = y^{-2} \longrightarrow \frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$

$$y^{-3}y' - y^{-2} = -e^{2x}$$

Mult. by -2

$$-2y^{-3}\frac{dy}{dx} + 2y^{-2} = 2e^{2x}$$

$$\frac{du}{dx} + 2u = 2e^{2x}$$

1st order linear

$$P_1(x) = 2$$

Int. factor: $\mu = e^{\int P_1(x)\,dx} = e^{\int 2\,dx} = e^{2x}$

$$e^{2x}(u' + 2u) = e^{2x}(2e^{2x})$$

$$\frac{d}{dx}\left[e^{2x}u\right] = 2e^{4x}$$

$$\int \frac{d}{dx}\left[e^{2x}u\right]dx = \int 2e^{4x}\,dx$$

$$e^{2x}u = \frac{2}{4}e^{4x} + C$$

$$u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}} = \frac{1}{2}e^{2x} + Ce^{-2x}$$

Let's go back to y so we can apply the I.C.

$$u = y^{-2} \Rightarrow y = u^{-\frac{1}{2}} = \frac{1}{\sqrt{u}}$$

Hence $y = \dfrac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}$.

We'll use $y(0)=1$ to find C.

$$y(0) = \frac{1}{\sqrt{\frac{1}{2}e^{0} + Ce^{0}}} = 1$$

$$\frac{1}{\sqrt{\frac{1}{2} + C}} = 1$$

$$\sqrt{\tfrac{1}{2} + C} = 1$$

$$\tfrac{1}{2} + C = 1^2 = 1$$

$$C = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

So the soln to the IVP

$$\boxed{y = \frac{1}{\sqrt{\tfrac{1}{2} e^{2x} + \tfrac{1}{2} e^{-2x}}}}$$