August 30 Math 2306 sec. 51 Spring 2023

Section 4: First Order Equations: Linear

First Order Linear ODE in Standard Form

We are considering first order ODEs of the form[a]

$$\frac{dy}{dx} + P(x)y = f(x), \tag{1}$$

where we'll assume that P and f are continuous on the interval of interest.

[a]The identifying characteristic that makes this *standard* form is that the coefficient of the highest derivative is 1.

The Solutions of $\frac{dy}{dx} + P(x)y = f(x)$

The solution to a first order linear ODE always has the same basic structure

$$y(x) = y_c(x) + y_p(x) \quad \text{where}$$

- $\boxed{y_c}$ is called the **complementary** solution. The complementary solution solves **associated homogeneous** equation, $\frac{dy}{dx} + P(x)y = 0$, and
- $\boxed{y_p}$ is called the **particular** solution. The particular solution depends heavily on f and is zero if $f(x) = 0$.

With higher order equations, we'll have to find y_c and y_p separately, but for first order equations we have a process for finding the whole solution.

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

To solve this equation, we seek a function $\mu(x)$ called an **integrating factor.** The idea is that multiplying the equation through by this function μ will result in the left side collapsing as the derivative of the product, $\frac{d}{dx}(\mu y)$. In other words, for the correct function μ

$$\mu\left(\frac{dy}{dx} + P(x)y\right) = \mu f(x) \quad \Longrightarrow \quad \frac{d}{dx}(\mu y) = \mu(x)f(x).$$

We found that the correct integrating factor is

$$\mu = e^{\int P(x)\,dx}.$$

Integrating Factor

Integrating Factor

For the first order, linear ODE in standard form

$$\frac{dy}{dx} + P(x)y = f(x),$$

the integrating factor

$$\mu(x) = \exp\left(\int P(x)\,dx\right).$$

Let's finish finding the solution y to the ODE and then look at the process and examples.

$$\frac{dy}{dx} + P(x)y = f(x)$$

Use the integrating factor, $\mu = e^{\int P(x)\,dx}$, to determine the solutions to the ODE.

mult by μ

$$\underbrace{\mu\left(\frac{dy}{dx} + P(x)y\right)}_{} = \mu f(x)$$

$$\frac{d}{dx}(\mu y) = \mu(x) f(x)$$

Integrate

$$\int \frac{d}{dx}(\mu y)\,dx = \int \mu(x) f(x)\,dx$$

$$\mu y = \int \mu(x) f(x)\,dx$$

$$y = \frac{1}{\mu} \int \mu(x) f(x)\, dx$$

$$y = \frac{1}{\mu} \left(\int \mu(x) f(x)\, dx + C \right)$$

$$y = \underbrace{\frac{1}{\mu(x)} \int \mu(x) f(x)\, dx}_{y_p} + \underbrace{\frac{C}{\mu(x)}}_{y_c}$$

Solution Process 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx$$

$$y(x) = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Example

Solve the initial value problem

$$x\frac{dy}{dx} - y = 2x^2, \; x > 0 \quad y(1) = 5$$

The ODE is 1st order linear. In standard form it is

$$\frac{dy}{dx} - \frac{1}{x}y = 2x \qquad P(x) = \frac{-1}{x}$$

Build μ: $\mu = e^{\int P(x)\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x}$

We can take the "$+C$" here to be zero.

$$\mu = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

Mult. ODE by μ $\quad x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = x^{-1}(2x)$

$$\frac{d}{dx}\left(x^{-1}y\right) = 2$$

$$\int \frac{d}{dx}\left(x^{-1}y\right)dx = \int 2\, dx$$

$$x^{-1}y = 2x + C$$

here, $+C$ is mandatory!

$$y = \frac{2x+C}{x^{-1}} = 2x^2 + Cx$$

$y = 2x^2 + Cx$ is a 1-parameter family of solutions to the ODE

Apply $y(1) = 5$

$$y(1) = 2(1)^2 + C(1) = 5 \Rightarrow 2 + C = 5 \Rightarrow C = 3$$

The solution to the IVP is

$$y = 2x^2 + 3x$$

Is $x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = \frac{d}{dx}\left(x^{-1}y\right)$?

LHS $x^{-1}\frac{dy}{dx} - x^{-1}\frac{1}{x}y = x^{-1}\frac{dy}{dx} - x^{-2}y$

RHS $\frac{d}{dx}\left(x^{-1}y\right) = x^{-1}\frac{dy}{dx} + (-1x^{-2})y$

$= x^{-1}\frac{dy}{dx} - x^{-2}y$

Identical!

Verify

Just for giggles, lets verify that our solution $y = 2x^2 + 3x$ really does solve the differential equation we started with

$$x\frac{dy}{dx} - y = 2x^2.$$

$y = 2x^2 + 3x, \quad y' = 4x + 3$

$x\frac{dy}{dx} - y \stackrel{?}{=} 2x^2$

$x(4x+3) - (2x^2 + 3x) \stackrel{?}{=} 2x^2$

$4x^2 + \cancel{3x} - 2x^2 - \cancel{3x} \stackrel{?}{=} 2x^2$

$2x^2 = 2x^2 \quad \checkmark$

Steady and Transient States

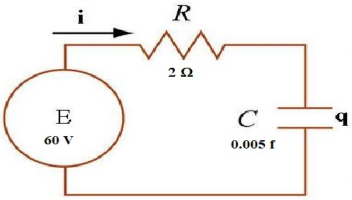

Figure: The charge $q(t)$ on the capacitor in the given curcuit satisfies a first order linear equation.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0.$$

Solve this IVP for the charge $q(t)$ on the capacitor for $t > 0$.

In standard form

$$\frac{dq}{dt} + 100q = 30$$

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0$$

$P(t) = 100$ Build μ

$\mu = e^{\int P(t)\,dt} = e^{\int 100\,dt} = e^{100t}$

$*$ e^{100t+C}

$e^{C} e^{100t}$

mult by μ $e^{100t}\left(\frac{dq}{dt} + 100q\right) = 30e^{100t}$

$\frac{d}{dt}\left(e^{100t} q\right) = 30e^{100t}$

$\int \frac{d}{dt}\left(e^{100t} q\right) dt = \int 30 e^{100t}\,dt$

$e^{100t} q = 30 \frac{e^{100t}}{100} + k$

$e^{100t} q = \frac{3}{10} e^{100t} + k$

$$q = \frac{\frac{3}{10}e^{100t} + k}{e^{100t}} = \frac{3}{10} + ke^{-100t}$$

1 parameter family of solutions is $q(t) = \frac{3}{10} + ke^{-100t}$

Apply $q(0)=0$ $\quad q(0) = \frac{3}{10} + ke^{-100(0)} = 0$

$$\frac{3}{10} + k = 0 \Rightarrow k = \frac{-3}{10}$$

The charge on the capacitor is

$$\boxed{q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}}$$

$$e^{100t}\left(\frac{dq}{dt} + 100q\right) \stackrel{?}{=} \frac{d}{dt}\left(e^{100t}q\right)$$

LHS $\quad e^{100t}q' + 100e^{100t}q$

RHS $\quad \frac{d}{dt}\left(e^{100t}q\right) = e^{100t}q' + 100e^{100t}q$

Steady and Transient States

Note that the solution, the charge, consists of a complementary and a particular solution, $q = q_p + q_c$.

$$q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}$$

$$q_c(t) = -\frac{3}{10}e^{-100t} \quad \text{and} \quad q_p(t) = \frac{3}{10}$$

Evaluate the limit

$$\lim_{t\to\infty} q_c(t) = \lim_{t\to\infty} \frac{-3}{10} e^{-100t} = 0$$

Steady and Transient States

The complementary solution contains the information given by the initial condition, and for some physical systems like this the complementary solution decays.

Definition: Such a decaying complementary solution is called a **transient state**.

Note that due to this decay, after a while $q(t) \approx q_p(t)$.

Definition: Such a corresponding particular solution is called a **steady state**.