August 31 Math 2306 sec. 51 Fall 2022

Section 4: First Order Equations: Linear

Recall that a first order linear ODE in standard form is

$$\frac{dy}{dx} + P(x)y = f(x).$$

If $f(x) = 0$ we call the equation homogeneous. Otherwise it's called nonhomogeneous. The general solution[1] will look like

$$y = y_c + y_p$$

where y_c is called the complementary solution and y_p is called the particular solution.

[1] If the ODE is homogeneous, the particular solution will not appear—it will be zero.

Solution Process for 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and divide out μ to solve for y.

Steady and Transient States

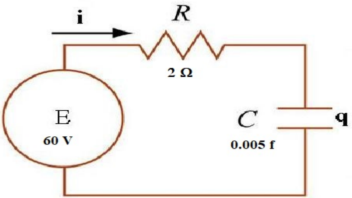

Figure: The charge $q(t)$ on the capacitor in the given curcuit satisfies a first order linear equation.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0.$$

Determine the charge $q(t)$ for $t > 0$.

Standard form: $\frac{dq}{dt} + 100q = 30$

$$P(t) = 100$$

$$\mu = e^{\int P(t)\,dt} = e^{\int 100\,dt} = e^{100t}$$

Multiply the ODE by μ

$$\frac{d}{dt}\left[e^{100t} q\right] = 30\,e^{100t}$$

Integrate

$$\int \frac{d}{dt}\left(e^{100t} q\right) dt = \int 30\,e^{100t}\,dt$$

$$e^{100t} q = 30\left(\frac{1}{100}\right) e^{100t} + k$$

$$\Rightarrow \quad q(t) = \frac{\frac{3}{10} e^{100t} + k}{e^{100t}}$$

$$q(t) = \frac{3}{10} + k e^{-100t}$$ This is the general solution.

Apply the IC $q(0) = 0$

$$q(0) = \frac{3}{10} + k e^{0} = 0 \quad \Rightarrow \quad k = \frac{-3}{10}$$

The charge on the capacitor is

$$\boxed{q(t) = \frac{3}{10} - \frac{3}{10} e^{-100t}}$$

Steady and Transient States

Note that the solution, the charge, consists of a complementary and a particular solution, $q = q_p + q_c$.

$$q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}$$

$$q_c(t) = -\frac{3}{10}e^{-100t} \quad \text{and} \quad q_p(t) = \frac{3}{10}$$

Evaluate the limit

$$\lim_{t\to\infty} q_c(t) = \lim_{t\to\infty} \frac{-3}{10} e^{-100t} = 0$$

Steady and Transient States

The complementary solution contains the information given by the initial condition, and for some physical systems like this the complementary solution decays.

Definition: Such a decaying complementary solution is called a **transient state**.

Note that due to this decay, after a while $q(t) \approx q_p(t)$.

Definition: Such a corresponding particular solution is called a **steady state**.

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a,b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx}+P(x)y=f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n\neq 0,1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Let $u = y^{1-n}$. u will solve a linear ODE.

$$u = y^{1-n}, \quad \frac{du}{dx} = (1-n)\,y^{1-n-1}\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1-n}\,y^n\,\frac{du}{dx}$$

sub this into the ODE

$$\frac{1}{1-n}\,y^n\,\frac{du}{dx} + P(x)\,y = f(x)\,y^n$$

Divide by y^n

$$\frac{1}{1-n}\frac{du}{dx} + P(x)\underbrace{\frac{y}{y^n}}_{y^{1-n} = u} = f(x)\underbrace{\frac{y^n}{y^n}}_{1}$$

$$\frac{1}{1-n}\frac{du}{dx} + P(x)u = f(x)$$

In standard form

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$$

$$\frac{du}{dx} + P_1(x)\,u = f_1(x)$$

where $P_1(x) = (1-n)P(x)$

$f_1(x) = (1-n)f(x)$

Since $u = y^{1-n}$, $y = u^{\frac{1}{1-n}}$

Example

$y' + P(x)\, y = f(x)\, y^n$

Solve the first order ODE $\dfrac{dy}{dx} - y = e^{-2x} y^3$.

This is Bernoulli with

$n = 3, \quad P(x) = -1, \quad f(x) = e^{-2x}$

$u = y^{1-n} = y^{1-3} = y^{-2}, \quad 1-n = -2$

u solves $\dfrac{du}{dx} + (1-n)P(x)\, u = (1-n) f(x)$

$\dfrac{du}{dx} + (-2)(-1)\, u = (-2)\, e^{-2x}$

$$\frac{du}{dx} + 2u = -2e^{-2x}$$

$$P_1(x) = 2, \qquad \mu = e^{\int P_1(x)dx} = e^{\int 2dx} = e^{2x}$$

Multiply by μ to get

$$\frac{d}{dx}\left(e^{2x}u\right) = e^{2x}\left(-2e^{-2x}\right)$$

$$\frac{d}{dx}\left(e^{2x}u\right) = -2$$

$$\int \frac{d}{dx}\left(e^{2x}u\right)dx = \int -2\,dx$$

$$e^{2x} u = -2x + C$$

$$u = \frac{-2x + C}{e^{2x}} = -2x e^{-2x} + C e^{-2x}$$

$$\boxed{\text{That is, } \quad u = C e^{-2x} - 2x e^{-2x}}$$

Now, go back to y.

$$u = y^{-2} \implies y = u^{-1/2} = \frac{1}{\sqrt{u}}$$

$$\boxed{\text{Hence} \quad y = \frac{1}{\sqrt{C e^{-2x} - 2x e^{-2x}}}.}$$