August 31 Math 2306 sec. 52 Fall 2022

Section 4: First Order Equations: Linear

Recall that a first order linear ODE in standard form is

$$\frac{dy}{dx} + P(x)y = f(x).$$

If $f(x) = 0$ we call the equation homogeneous. Otherwise it's called nonhomogeneous. The general solution[1] will look like

$$y = y_c + y_p$$

where y_c is called the complementary solution and y_p is called the particular solution.

[1] If the ODE is homogeneous, the particular solution will not appear—it will be zero.

Solution Process for 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and divide out μ to solve for y.

Steady and Transient States

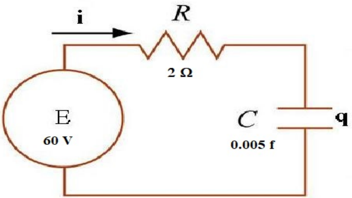

Figure: The charge $q(t)$ on the capacitor in the given curcuit satisfies a first order linear equation.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0.$$

Determine the charge $q(t)$ for $t > 0$.

In standard form:

$$\frac{dq}{dt} + 100q = 30$$

$P(t) = 100$; $\mu = e^{\int p(t)\,dt} = e^{\int 100\,dt} = e^{100t}$

Multiply the ODE by μ

$$\frac{d}{dt}\left(e^{100t} q\right) = 30e^{100t}$$

$$\int \frac{d}{dt}\left(e^{100t} q\right) dt = \int 30 e^{100t} dt$$

$$e^{100t} q = 30\left(\frac{1}{100}\right) e^{100t} + k$$

Divide by μ $\quad q = \dfrac{\frac{3}{10} e^{100t} + k}{e^{100t}}$

The general solution is

$$q = \frac{3}{10} + k e^{-100t}$$

Apply $q(0) = 0$

$$q(0) = \frac{3}{10} + k e^{0} = 0 \Rightarrow k = \frac{-3}{10}$$

$$\boxed{\text{The charge on the capacitor is} \quad q(t) = \frac{3}{10} - \frac{3}{10} e^{-100t}}$$

Steady and Transient States

Note that the solution, the charge, consists of a complementary and a particular solution, $q = q_p + q_c$.

$$q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}$$

$$q_c(t) = -\frac{3}{10}e^{-100t} \quad \text{and} \quad q_p(t) = \frac{3}{10}$$

Evaluate the limit

$$\lim_{t\to\infty} q_c(t) = \lim_{t\to\infty} \frac{-3}{10} e^{-100t} = 0$$

Steady and Transient States

The complementary solution contains the information given by the initial condition, and for some physical systems like this the complementary solution decays.

Definition: Such a decaying complementary solution is called a **transient state**.

Note that due to this decay, after a while $q(t) \approx q_p(t)$.

Definition: Such a corresponding particular solution is called a **steady state**.

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a,b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

We'll do a change of variables

Set $u = y^{1-n}$. u will solve a linear ODE

$$\frac{du}{dx} = (1-n)\underbrace{y^{1-n-1}}_{y^{-n}}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{1-n}y^n\frac{du}{dx}.$$

Sub this into the ODE

$$\frac{1}{1-n}y^n\frac{du}{dx} + P(x)y = f(x)y^n$$

Divide by y^n

$$\frac{1}{1-n}\frac{du}{dx} + P(x)\underbrace{\frac{y}{y^n}}_{y^{1-n} = u} = f(x)\underbrace{\frac{y^n}{y^n}}_{1}$$

$$\frac{1}{1-n}\frac{du}{dx} + P(x)\,u = f(x)$$

1st order linear ODE

In standard form

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)f(x)$$

$$\frac{du}{dx} + P_1(x)\,u = f_1(x)$$

where $P_1(x) = (1-n)\,P(x)$

$f_1(x) = (1-n)\,f(x)$

From $u = y^{1-n}$, $y = u^{\frac{1}{1-n}}$

Example

Solve the first order ODE $\dfrac{dy}{dx} - y = e^{-2x}y^3$.

$$\frac{dy}{dx} + P(x)\,y = f(x)\,y^n$$

This is Bernoulli w/

$$P(x) = -1, \quad f(x) = e^{-2x}, \quad n = 3$$

$$u = y^{1-n} = y^{1-3} = y^{-2}$$

u solves $\quad \dfrac{du}{dx} + (1-n)P(x)\,u = (1-n)f(x)$

$$\frac{du}{dx} + (-2)(-1)\,u = (-2)\,e^{-2x}$$

$$\frac{du}{dx} + 2u = -2e^{-2x}$$

This is 1st order linear in standard form.

$P_1(x) = 2$, $\quad \mu = e^{\int P_1(x)dx} = e^{\int 2\,dx} = e^{2x}$

$$e^{2x}\left(\frac{du}{dx} + 2u\right) = e^{2x}\left(-2e^{-2x}\right)$$

$$\frac{d}{dx}\left(e^{2x}u\right) = -2$$

$$\int \frac{d}{dx}\left(e^{2x}u\right)dx = \int -2\,dx$$

$$e^{2x}u = -2x + C$$

$$u = \frac{-2x + C}{e^{2x}} = e^{-2x}(-2x + C)$$

From $u = y^{-2}$, $y = u^{-1/2} = \frac{1}{\sqrt{u}}$

The solution

$$\boxed{y = \frac{1}{\sqrt{e^{-2x}(c - 2x)}}}$$

$$e^{2x}\left(\frac{du}{dx}+2u\right) = e^{2x}\frac{du}{dx} + 2e^{2x}u$$

$$\frac{d}{dx}\left(e^{2x}u\right) = e^{2x}\frac{du}{dx} + 2e^{2x}u$$