Section 8.3 *Integral and Comparison Tests*

APEX CALCULUS II

Version 2.0

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Contents

8.3 Integral and Comparison Tests

Knowing whether or not a series converges is very important, especially when we discuss Power Series in Section 8.6. Theorems 60 and 61 give criteria for when Geometric and p- series converge, and Theorem 63 gives a quick test to determine if a series diverges. There are many important series whose convergence cannot be determined by these theorems, though, so we introduce a set of tests that allow us to handle a broad range of series. We start with the Integral Test.

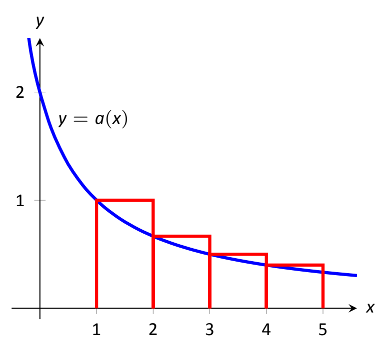
Integral Test

We stated in Section 8.1 that a sequence $\{a_n\}$ is a function $a(n)$ whose domain is $\mathbb{N}$, the set of natural numbers. If we can extend $a(n)$ to $\mathbb{R}$, the real numbers, and it is both positive and decreasing on $[1, \infty)$, then the convergence of $\sum_{n=1}^{\infty} a_n$ is the same as $\int_1^{\infty} a(x)\, dx$.

Note: Theorem 65 does not state that the integral and the summation have the same value.

Theorem 65 Integral Test

Let a sequence $\{a_n\}$ be defined by $a_n = a(n)$, where $a(n)$ is continuous, positive and decreasing on $[1, \infty)$. Then $\sum_{n=1}^{\infty} a_n$ converges, if, and only if, $\int_1^{\infty} a(x)\, dx$ converges.



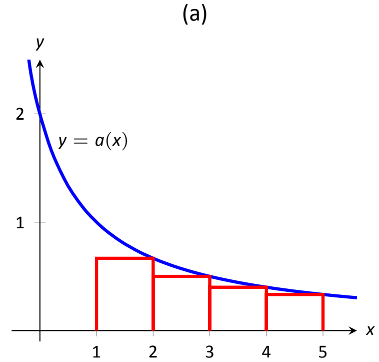


Figure 8.15: Illustrating the truth of the Integral Test.

We can demonstrate the truth of the Integral Test with two simple graphs. In Figure 8.15(a), the height of each rectangle is $a(n) = a_n$ for $n = 1, 2, \ldots$, and clearly the rectangles enclose more area than the area under $y = a(x)$. Therefore we can conclude that

$$\int_1^{\infty} a(x)\, dx < \sum_{n=1}^{\infty} a_n. \tag{8.1}$$

In Figure 8.15(b), we draw rectangles under $y = a(x)$ with the Right-Hand rule, starting with $n = 2$. This time, the area of the rectangles is less than the area under $y = a(x)$, so $\sum_{n=2}^{\infty} a_n < \int_1^{\infty} a(x)\, dx$. Note how this summation starts with $n = 2$; adding a_1 to both sides lets us rewrite the summation starting with

Notes:

$n = 1$:

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx. \tag{8.2}$$

Combining Equations (8.1) and (8.2), we have

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx < a_1 + \sum_{n=1}^{\infty} a_n. \tag{8.3}$$

From Equation (8.3) we can make the following two statements:

1. If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\int_1^{\infty} a(x)\,dx$ (because $\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,dx$)
2. If $\sum_{n=1}^{\infty} a_n$ converges, so does $\int_1^{\infty} a(x)\,dx$ (because $\int_1^{\infty} a(x)\,dx < \sum_{n=1}^{\infty} a_n$.)

Therefore the series and integral either both converge or both diverge. Theorem 64 allows us to extend this theorem to series where a_n is positive and decreasing on $[b, \infty)$ for some $b > 1$.

Example 241 **Using the Integral Test**

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. (The terms of the sequence $\{a_n\} = \{\ln n/n^2\}$ and the n^{th} partial sums are given in Figure 8.16.)

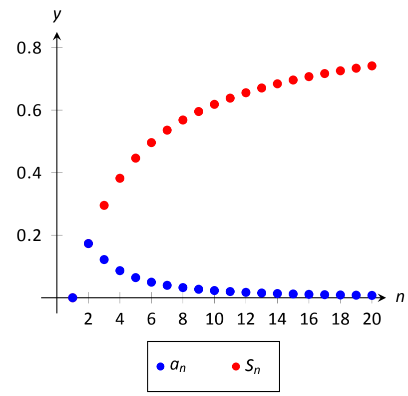


Figure 8.16: Plotting the sequence and series in Example 241.

Solution Applying the Integral Test, we test the convergence of $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$. Integrating this improper integral requires the use of Integration by Parts, with $u = \ln x$ and $dv = 1/x^2\,dx$.

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{x}\ln x\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{x}\ln x - \frac{1}{x}\Big|_1^b \\
&= \lim_{b\to\infty} 1 - \frac{1}{b} - \frac{\ln b}{b}. \quad \text{Apply L'Hôpital's Rule:} \\
&= 1.
\end{aligned}$$

Since $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ converges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

Notes:

Note how the sequence $\{a_n\}$ is not strictly decreasing; it increases from $n = 1$ to $n = 2$. However, this does not keep us from applying the Integral Test as the sequence in positive and decreasing on $[2, \infty)$.

Theorem 61 was given without justification, stating that the general p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$. In the following example, we prove this to be true by applying the Integral Test.

Example 242 **Using the Integral Test to establish Theorem 61.**

Use the Integral Test to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

Solution Consider the integral $\displaystyle\int_1^{\infty} \frac{1}{(ax+b)^p}\,dx$; assuming $p \neq 1$,

$$\begin{aligned}
\int_1^{\infty} \frac{1}{(ax+b)^p}\,dx &= \lim_{c\to\infty} \int_1^{c} \frac{1}{(ax+b)^p}\,dx \\
&= \lim_{c\to\infty} \frac{1}{a(1-p)}(ax+b)^{1-p}\Big|_1^c \\
&= \lim_{c\to\infty} \frac{1}{a(1-p)}\big((ac+b)^{1-p} - (a+b)^{1-p}\big).
\end{aligned}$$

This limit converges if, and only if, $p > 1$. It is easy to show that the integral also diverges in the case of $p = 1$. (This result is similar to the work preceding Key Idea 21.)

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$.

We consider two more convergence tests in this section, both *comparison* tests. That is, we determine the convergence of one series by comparing it to another series with known convergence.

Notes:

Direct Comparison Test

Theorem 66 Direct Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences where $a_n \leq b_n$ for all $n \geq N$, for some $N \geq 1$.

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

Note: A sequence $\{a_n\}$ is a **positive sequence** if $a_n > 0$ for all n.

Because of Theorem 64, any theorem that relies on a positive sequence still holds true when $a_n > 0$ for all but a finite number of values of n.

Example 243 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$.

Solution This series is neither a geometric or p-series, but seems related. We predict it will converge, so we look for a series with larger terms that converges. (Note too that the Integral Test seems difficult to apply here.)

Since $3^n < 3^n + n^2$, $\dfrac{1}{3^n} > \dfrac{1}{3^n + n^2}$ for all $n \geq 1$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series; by Theorem 66, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$ converges.

Example 244 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n - \ln n}$.

Solution We know the Harmonic Series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and it seems that the given series is closely related to it, hence we predict it will diverge.

Since $n \geq n - \ln n$ for all $n \geq 1$, $\dfrac{1}{n} \leq \dfrac{1}{n - \ln n}$ for all $n \geq 1$.

The Harmonic Series diverges, so we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n - \ln n}$ diverges as well.

Notes:

The concept of direct comparison is powerful and often relatively easy to apply. Practice helps one develop the necessary intuition to quickly pick a proper series with which to compare. However, it is easy to construct a series for which it is difficult to apply the Direct Comparison Test.

Consider $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+\ln n}$. It is very similar to the divergent series given in Example 244. We suspect that it also diverges, as $\dfrac{1}{n}\approx\dfrac{1}{n+\ln n}$ for large n. However, the inequality that we naturally want to use "goes the wrong way": since $n \le n+\ln n$ for all $n \ge 1$, $\dfrac{1}{n}\ge\dfrac{1}{n+\ln n}$ for all $n \ge 1$. The given series has terms *less than* the terms of a divergent series, and we cannot conclude anything from this.

Fortunately, we can apply another test to the given series to determine its convergence.

Limit Comparison Test

Theorem 67 Limit Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

1. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=L$, where L is a positive real number, then $\displaystyle\sum_{n=1}^{\infty}a_n$ and $\displaystyle\sum_{n=1}^{\infty}b_n$ either both converge or both diverge.
2. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=0$, then if $\displaystyle\sum_{n=1}^{\infty}b_n$ converges, then so does $\displaystyle\sum_{n=1}^{\infty}a_n$.
3. If $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=\infty$, then if $\displaystyle\sum_{n=1}^{\infty}b_n$ diverges, then so does $\displaystyle\sum_{n=1}^{\infty}a_n$.

It is helpful to remember that when using Theorem 67, the terms of the series with known convergence go in the denominator of the fraction.

We use the Limit Comparison Test in the next example to examine the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+\ln n}$ which motivated this new test.

Notes:

Example 245 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ using the Limit Comparison Test.

Solution We compare the terms of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ to the terms of the Harmonic Sequence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/(n+\ln n)}{1/n} &= \lim_{n\to\infty} \frac{n}{n+\ln n}\\ &= 1 \quad \text{(after applying L'Hôpital's Rule).}\end{aligned}$$

Since the Harmonic Series diverges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ diverges as well.

Example 246 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n-n^2}$

Solution This series is similar to the one in Example 243, but now we are considering "$3^n - n^2$" instead of "$3^n + n^2$." This difference makes applying the Direct Comparison Test difficult.

Instead, we use the Limit Comparison Test and compare with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/(3^n-n^2)}{1/3^n} &= \lim_{n\to\infty} \frac{3^n}{3^n-n^2}\\ &= 1 \quad \text{(after applying L'Hôpital's Rule twice).}\end{aligned}$$

We know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series, hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n-n^2}$ converges as well.

As mentioned before, practice helps one develop the intuition to quickly choose a series with which to compare. A general rule of thumb is to pick a series based on the dominant term in the expression of $\{a_n\}$. It is also helpful to note that factorials dominate exponentials, which dominate algebraic functions (e.g., polynomials), which dominate logarithms. In the previous example,

Notes:

the dominant term of $\dfrac{1}{3^n - n^2}$ was 3^n, so we compared the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. It is hard to apply the Limit Comparison Test to series containing factorials, though, as we have not learned how to apply L'Hôpital's Rule to $n!$.

Example 247 **Applying the Limit Comparison Test**

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{x}+3}{x^2 - x + 1}$.

Solution We naïvely attempt to apply the rule of thumb given above and note that the dominant term in the expression of the series is $1/x^2$. Knowing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we attempt to apply the Limit Comparison Test:

$$\begin{aligned} \lim_{n\to\infty} \frac{(\sqrt{x}+3)/(x^2-x+1)}{1/x^2} &= \lim_{n\to\infty} \frac{x^2(\sqrt{x}+3)}{x^2-x+1} \\ &= \infty \quad \text{(Apply L'Hôpital's Rule).} \end{aligned}$$

Theorem 67 part (3) only applies when $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges; in our case, it converges. Ultimately, our test has not revealed anything about the convergence of our series.

The problem is that we chose a poor series with which to compare. Since the numerator and denominator of the terms of the series are both algebraic functions, we should have compared our series to the dominant term of the numerator divided by the dominant term of the denominator.

The dominant term of the numerator is $x^{1/2}$ and the dominant term of the denominator is x^2. Thus we should compare the terms of the given series to $x^{1/2}/x^2 = 1/x^{3/2}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{(\sqrt{x}+3)/(x^2-x+1)}{1/x^{3/2}} &= \lim_{n\to\infty} \frac{x^{3/2}(\sqrt{x}+3)}{x^2-x+1} \\ &= 1 \quad \text{(Applying L'Hôpital's Rule).} \end{aligned}$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^{3/2}}$ converges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{x}+3}{x^2-x+1}$ converges as well.

Notes:

Exercises 8.3

Terms and Concepts

1. In order to apply the Integral Test to a sequence $\{a_n\}$, the function $a(n) = a_n$ must be _____, _____ and _____.
2. T/F: The Integral Test can be used to determine the sum of a convergent series.
3. What test(s) in this section do not work well with factorials?
4. Suppose $\sum_{n=0}^{\infty} a_n$ is convergent, and there are sequences $\{b_n\}$ and $\{c_n\}$ such that $b_n \leq a_n \leq c_n$ for all n. What can be said about the series $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$?

Problems

In Exercises 5 – 12, use the Integral Test to determine the convergence of the given series.

5. $\sum_{n=1}^{\infty} \frac{1}{2^n}$
6. $\sum_{n=1}^{\infty} \frac{1}{n^4}$
7. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$
8. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$
9. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
10. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
11. $\sum_{n=1}^{\infty} \frac{n}{2^n}$
12. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

In Exercises 13 – 22, use the Direct Comparison Test to determine the convergence of the given series; state what series is used for comparison.

13. $\sum_{n=1}^{\infty} \frac{1}{n^2+3n-5}$
14. $\sum_{n=1}^{\infty} \frac{1}{4^n+n^2-n}$
15. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$
16. $\sum_{n=1}^{\infty} \frac{1}{n!+n}$
17. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$
18. $\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}-2}$
19. $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-5}$
20. $\sum_{n=1}^{\infty} \frac{2^n}{5^n+10}$
21. $\sum_{n=2}^{\infty} \frac{n}{n^2-1}$
22. $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$

In Exercises 23 – 32, use the Limit Comparison Test to determine the convergence of the given series; state what series is used for comparison.

23. $\sum_{n=1}^{\infty} \frac{1}{n^2-3n+5}$
24. $\sum_{n=1}^{\infty} \frac{1}{4^n-n^2}$
25. $\sum_{n=4}^{\infty} \frac{\ln n}{n-3}$
26. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$
27. $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$
28. $\sum_{n=1}^{\infty} \frac{n-10}{n^2+10n+10}$
29. $\sum_{n=1}^{\infty} \sin\left(1/n\right)$
30. $\sum_{n=1}^{\infty} \frac{n+5}{n^3-5}$
31. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2+17}$
32. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+100}$

In Exercises 33 – 40, determine the convergence of the given series. State the test used; more than one test may be appropriate.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+5)^3}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n!}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+n}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{10n+5}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{\sqrt{n}}$

41. Given that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, state which of the following series converges, may converge, or does not converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} a_n a_{n+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$

(d) $\displaystyle\sum_{n=1}^{\infty} n a_n$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$

41. Left to reader

Section 8.2

1. Answers will vary.
3. One sequence is the sequence of terms $\{a_n\}$. The other is the sequence of n^{th} partial sums, $\{S_n\} = \{\sum_{i=1}^{n} a_i\}$.
5. F
7. (a) $1, \frac{5}{4}, \frac{49}{36}, \frac{205}{144}, \frac{5269}{3600}$
   (b) Plot omitted
9. (a) $1, 3, 6, 10, 15$
   (b) Plot omitted
11. (a) $\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \frac{121}{243}$
   (b) Plot omitted
13. (a) $0.1, 0.11, 0.111, 0.1111, 0.11111$
   (b) Plot omitted
15. $\lim\limits_{n\to\infty} a_n = \infty$; by Theorem 63 the series diverges.
17. $\lim\limits_{n\to\infty} a_n = 1$; by Theorem 63 the series diverges.
19. $\lim\limits_{n\to\infty} a_n = e$; by Theorem 63 the series diverges.
21. Converges
23. Converges
25. Converges
27. Converges
29. Diverges
31. (a) $S_n = \left(\frac{n(n+1)}{2}\right)^2$
   (b) Diverges
33. (a) $S_n = 5\frac{1-1/2^n}{1/2}$
   (b) Converges to 10.
35. (a) $S_n = \frac{1-(-1/3)^n}{4/3}$
   (b) Converges to $3/4$.
37. (a) With partial fractions, $a_n = \frac{3}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$. Thus $S_n = \frac{3}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$.
   (b) Converges to $9/4$
39. (a) $S_n = \ln\left(1/(n+1)\right)$
   (b) Diverges (to $-\infty$).
41. (a) $a_n = \frac{1}{n(n+3)}$; using partial fractions, the resulting telescoping sum reduces to $S_n = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$
   (b) Converges to $11/18$.
43. (a) With partial fractions, $a_n = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$. Thus $S_n = \frac{1}{2}\left(3/2 - \frac{1}{n} - \frac{1}{n+1}\right)$.
   (b) Converges to $3/4$.
45. (a) The n^{th} partial sum of the odd series is $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. The n^{th} partial sum of the even series is $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}$. Each term of the even series is less than the corresponding term of the odd series, giving us our result.
   (b) The n^{th} partial sum of the odd series is $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. The n^{th} partial sum of 1 plus the even series is $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2(n-1)}$. Each term of the even series is now greater than or equal to the corresponding term of the odd series, with equality only on the first term. This gives us the result.
   (c) If the odd series converges, the work done in (a) shows the even series converges also. (The sequence of the n^{th} partial sum of the even series is bounded and monotonically increasing.) Likewise, (b) shows that if the even series converges, the odd series will, too. Thus if either series converges, the other does.

   Similarly, (a) and (b) can be used to show that if either series diverges, the other does, too.
   (d) If both the even and odd series converge, then their sum would be a convergent series. This would imply that the Harmonic Series, their sum, is convergent. It is not. Hence each series diverges.

Section 8.3

1. continuous, positive and decreasing
3. The Integral Test (we do not have a continuous definition of $n!$ yet) and the Limit Comparison Test (same as above, hence we cannot take its derivative).
5. Converges
7. Diverges
9. Converges
11. Converges
13. Converges; compare to $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, as $1/(n^2+3n-5) \leq 1/n^2$ for all $n > 1$.
15. Diverges; compare to $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, as $1/n \leq \ln n/n$ for all $n \geq 2$.
17. Diverges; compare to $\sum\limits_{n=1}^{\infty} \frac{1}{n}$. Since $n = \sqrt{n^2} > \sqrt{n^2-1}$, $1/n \leq 1/\sqrt{n^2-1}$ for all $n \geq 2$.
19. Diverges; compare to $\sum\limits_{n=1}^{\infty} \frac{1}{n}$:
$$\frac{1}{n} = \frac{n^2}{n^3} < \frac{n^2+n+1}{n^3} < \frac{n^2+n+1}{n^3-5},$$
for all $n \geq 1$.
21. Diverges; compare to $\sum\limits_{n=1}^{\infty} \frac{1}{n}$. Note that
$$\frac{n}{n^2-1} = \frac{n^2}{n^2-1} \cdot \frac{1}{n} > \frac{1}{n},$$
as $\frac{n^2}{n^2-1} > 1$, for all $n \geq 2$.
23. Converges; compare to $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.
25. Diverges; compare to $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n}$.
27. Diverges; compare to $\sum\limits_{n=1}^{\infty} \frac{1}{n}$.

29. Diverges; compare to $\sum_{n=1}^{\infty} \frac{1}{n}$. Just as $\lim_{n\to 0} \frac{\sin n}{n} = 1$, $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$.

31. Converges; compare to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

33. Converges; Integral Test

35. Diverges; the n^{th} Term Test and Direct Comparison Test can be used.

37. Converges; the Direct Comparison Test can be used with sequence $1/3^n$.

39. Diverges; the n^{th} Term Test can be used, along with the Integral Test.

41. (a) Converges; use Direct Comparison Test as $\frac{a_n}{n} < n$.
    (b) Converges; since original series converges, we know $\lim_{n\to\infty} a_n = 0$. Thus for large n, $a_n a_{n+1} < a_n$.
    (c) Converges; similar logic to part (b) so $(a_n)^2 < a_n$.
    (d) May converge; certainly $na_n > a_n$ but that does not mean it does not converge.
    (e) Does not converge, using logic from (b) and n^{th} Term Test.

Section 8.4

1. algebraic, or polynomial.

3. Integral Test, Limit Comparison Test, and Root Test

5. Converges

7. Converges

9. The Ratio Test is inconclusive; the p-Series Test states it diverges.

11. Converges

13. Converges; note the summation can be rewritten as $\sum_{n=1}^{\infty} \frac{2^n n!}{3^n n!}$, from which the Ratio Test can be applied.

15. Converges

17. Converges

19. Diverges

21. Diverges. The Root Test is inconclusive, but the n^{th}-Term Test shows divergence. (The terms of the sequence approach e^2, not 0, as $n \to \infty$.)

23. Converges

25. Diverges; Limit Comparison Test

27. Converges; Ratio Test or Limit Comparison Test with $1/3^n$.

29. Diverges; n^{th}-Term Test or Limit Comparison Test with 1.

31. Diverges; Direct Comparison Test with $1/n$

33. Converges; Root Test

Section 8.5

1. The signs of the terms do not alternate; in the given series, some terms are negative and the others positive, but they do not necessarily alternate.

3. Many examples exist; one common example is $a_n = (-1)^n/n$.

5. (a) converges
   (b) converges (p-Series)
   (c) absolute

7. (a) diverges (limit of terms is not 0)
   (b) diverges
   (c) n/a; diverges

9. (a) converges
   (b) diverges (Limit Comparison Test with $1/n$)
   (c) conditional

11. (a) diverges (limit of terms is not 0)
    (b) diverges
    (c) n/a; diverges

13. (a) diverges (terms oscillate between ± 1)
    (b) diverges
    (c) n/a; diverges

15. (a) converges
    (b) converges (Geometric Series with $r = 2/3$)
    (c) absolute

17. (a) converges
    (b) converges (Ratio Test)
    (c) absolute

19. (a) converges
    (b) diverges (p-Series Test with $p = 1/2$)
    (c) conditional

21. $S_5 = -1.1906$; $S_6 = -0.6767$;
$$-1.1906 \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} \leq -0.6767$$

23. $S_6 = 0.3681$; $S_7 = 0.3679$;
$$0.3681 \leq \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \leq 0.3679$$

25. $n = 5$

27. Using the theorem, we find $n = 499$ guarantees the sum is within 0.001 of $\pi/4$. (Convergence is actually faster, as the sum is within ε of $\pi/24$ when $n \geq 249$.)

Section 8.6

1. 1

3. 5

5. $1 + 2x + 4x^2 + 8x^3 + 16x^4$

7. $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

9. (a) $R = \infty$
   (b) $(-\infty, \infty)$

11. (a) $R = 1$
    (b) $(2, 4]$

13. (a) $R = 2$
    (b) $(-2, 2)$

15. (a) $R = 1/5$
    (b) $(4/5, 6/5)$

17. (a) $R = 1$
    (b) $(-1, 1)$

19. (a) $R = \infty$
    (b) $(-\infty, \infty)$