December 1 Math 3260 sec. 53 Fall 2025

6.2 Eigenvalues & Eigenvectors

Definition

Let A be an $n \times n$ matrix. An **eigenvalue** of A is a scalar λ for which there exists a nonzero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}. \tag{1}$$

For a given eigenvalue λ, a nonzero vector $\vec{x}$ satisfying equation (1) is called an **eigenvector** corresponding to the eigenvalue λ.

The Characteristic Equation

Let A be an $n \times n$ matrix. The function

$$P_A(\lambda) = \det(A - \lambda I_n)$$

is called the **characteristic polynomial** of the matrix A. The equation

$$P_A(\lambda) = 0, \quad \text{i.e.,} \quad \det(A - \lambda I_n) = 0$$

is called the **characteristic equation** of the matrix A.

Theorem

Let A be an $n \times n$ matrix, and let $P_A(\lambda)$ be the characteristic polynomial of A. The number λ_0 is an eigenvalue of A if and only if $P_A(\lambda_0) = 0$. That is, λ_0 is an eigenvalue of A if and only if it is a root of the characteristic equation $\det(A - \lambda I_n) = 0$.

Example $A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

The characteristic polynomial was

$$P_A(\lambda) = -(3+\lambda)(\lambda-5)(\lambda-1) = -\lambda^3 + 3\lambda^2 + 13\lambda - 15.$$

Find an eigenvector for each eigenvalue.

Last time, we concluded that the three eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 5$ and $\lambda_3 = 1$. For $\lambda_2 = 5$, the matrix

$$A - 5I_3 = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -3 & 2 \\ 0 & 0 & -8 \end{bmatrix} \quad \text{with} \quad \operatorname{rref}(A - 5I_3) = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So a solution $\vec{x}$ to $(A - 5I_3)\vec{x} = \vec{0}_3$ would have entries, $x_1 = 3x_2$, x_2 is free, and $x_3 = 0$. The eigenvectors associated with $\lambda_2 = 5$ are

$$\vec{x} = t\langle 3, 1, 0\rangle, \quad t \neq 0.$$

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

Eigenspaces & Eigenbases

Definition

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The **eigenspace corresponding to the eigenvalue** λ_0 is the set

$$E_A(\lambda_0) = \left\{\vec{x} \in R^n \mid A\vec{x} = \lambda_0\vec{x}\right\} = \mathcal{N}(A - \lambda_0 I_n).$$

An **eigenspace** is a null space, so it's a subspace of R^n. We can find a basis the way we regularly find the basis for a null space.

An **eigenspace** is all of the eigenvectors for a given eigenvalue with the zero vector thrown in to make a subspace.

Example

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$. Find the characteristic polynomials $P_A(\lambda)$ and $P_B(\lambda)$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \quad P_A(\lambda) = P_B(\lambda) = (2-\lambda)^2(4-\lambda)$$

Find bases for the eigenspaces $E_A(2)$ and $E_B(2)$.

Two Types of Multiplicities

Geometric Multiplicity

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The dimension of the eigenspace, $\dim(E_A(\lambda_0))$, corresponding to λ_0 is called the **geometric multiplicity** of λ_0.

To determine the geometric multiplicity, we have to find the dimension of the eigenspace—i.e., how many free variables are there in the equation

$$(A - \lambda_0 I_n)\vec{x} = \vec{0}_n.$$

Algebraic Multiplicity

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The **algebraic multiplicity** of λ_0 is its multiplicity as the root of the characteristic equation $P_A(\lambda) = 0$. That is, if $(\lambda - \lambda_0)^k$ is a factor of $P_A(\lambda)$ and $(\lambda - \lambda_0)^{k+1}$ is not a factor of $P_A(\lambda)$, then the algebraic multiplicity of λ_0 is k.

If the characteristic polynomial was $(3-\lambda)^4(7-\lambda)^2(-2-\lambda)$, the eigenvalues with their algebraic multiplicities would be

$$\lambda = 3 \text{ algebraic multiplicity } 4,$$

$$\lambda = 7 \text{ algebraic multiplicity } 2,$$

$$\lambda = -2 \text{ algebraic multiplicity } 1.$$

The algebraic multiplicity is always greater than or equal to the geometric multiplicity.

Example

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P_A(\lambda) = (2-\lambda)^2(4-\lambda) \quad \text{and} \quad P_B(\lambda) = (2-\lambda)^2(4-\lambda).$$

Both have eigenvalue $\lambda = 2$ with **algebraic multiplicity** of two.

$$\underbrace{\{\langle 1,0,0\rangle, \langle 0,1,0\rangle\}}_{\text{basis for } E_A(2)} \qquad \underbrace{\{\langle 1,0,0\rangle\}}_{\text{basis for } E_B(2)}$$

$\lambda = 2$ has geometric multiplicity **two** as an eigenvalue of A and it has a geometric multiplicity **one** as an eigenvalue of B.

If a matrix has enough linearly independent eigenvectors, we may be able to build a basis for R^n out of eigenvectors. So the geometric multiplicity is of interest as is their linear independence.

Theorem

Let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ be a set of eigenvectors of an $n \times n$ matrix corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent.

Note

If A is an $n \times n$ matrix with n distinct eigenvalues, then A has a set of n linearly independent eigenvectors.

Definition

Let A be an $n \times n$ matrix. If A has n linearly independent eigenvectors, $\vec{x}_1, \ldots, \vec{x}_n$ (combined across all eigenvalues), then the set $\mathcal{E}_A = \{\vec{x}_1, \ldots, \vec{x}_n\}$ is a basis for R^n called an **eigenbasis** for A.

Suppose A is $n \times n$

- If A has n distinct eigenvalues, it is guaranteed to have an eigenbasis.
- If A has fewer than n distinct eigenvalues, then
  - it has an eigenbasis if the sum of all geometric multiplicities is n;
  - it doesn't have an eigenbasis if the sum of all geometric multiplicities is smaller than n.

Example

Find an eigenbasis for $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ or show that it is not possible.

Example: $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ $\quad \lambda_1 = 6 \quad \vec{x}_1 = \langle 1, 1 \rangle \quad \lambda_2 = -3 \quad \vec{x}_2 = \langle -8, 1 \rangle$

1. Create a matrix C having the eigenvectors as its column vectors.
2. Find C^{-1}.
3. Find the product $C^{-1}AC$.

6.3 Diagonalization

Definition

An $n \times n$ matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix. That is, A is diagonalizable if there exists a diagonal matrix D and an invertible matrix C such that

$$D = C^{-1}AC.$$

The previous example suggests that diagonalizability is related to making a matrix out of eigenvectors. This turns out to be true, but to get an $n \times n$ matrix that is actually invertible, we need n linearly independent vectors. This is where having an eigenbasis comes in.

Facts About Similar Matrices

Theorem

If A and B are similar matrices, the $\det(A) = \det(B)$.

Theorem

If A and B are similar matrices, then A and B have the same eigenvalues, each with the same algebraic and geometric multiplicities.

If A and B are similar, so they share an eigenvalue λ, the eigenvectors corresponding to λ are **generally different**.

$B = C^{-1}AC$

Show that $\det(B) = \det(A)$ and $P_B(\lambda) = P_A(\lambda)$.

Theorem

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

Moreover, if A is diagonalizable, then there exists a diagonal matrix D such that $D = C^{-1}AC$ where the columns of the invertible matrix C are the vectors in an eigenbasis, $\mathcal{E}_A$, for the matrix A, and the diagonal entries of the matrix D are the eigenvalues of A.

Remark: If A has n distinct eigenvalues, then it is guaranteed to be diagonalizable. If it has less than n distinct eigenvalues, it may or may not be diagonalizable. A is diagonalizable if the sum of the geometric multiplicities is n.

Example

Let $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. The characteristic polynomial $P_A(\lambda) = (1-\lambda)(2+\lambda)^2$. Determine whether A is diagonalizable.

Example

Diagonalize the matrix $A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$ if possible.

Evaluate A^{10} if $A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$.