December 1 Math 3260 sec. 53 Fall 2025

6.2 Eigenvalues & Eigenvectors

Definition

Let A be an $n \times n$ matrix. An **eigenvalue** of A is a scalar λ for which there exists a nonzero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}. \tag{1}$$

For a given eigenvalue λ, a nonzero vector $\vec{x}$ satisfying equation (1) is called an **eigenvector** corresponding to the eigenvalue λ.

The Characteristic Equation

Let A be an $n \times n$ matrix. The function

$$P_A(\lambda) = \det(A - \lambda I_n)$$

is called the **characteristic polynomial** of the matrix A. The equation

$$P_A(\lambda) = 0, \quad \text{i.e.,} \quad \det(A - \lambda I_n) = 0$$

is called the **characteristic equation** of the matrix A.

Theorem

Let A be an $n \times n$ matrix, and let $P_A(\lambda)$ be the characteristic polynomial of A. The number λ_0 is an eigenvalue of A if and only if $P_A(\lambda_0) = 0$. That is, λ_0 is an eigenvalue of A if and only if it is a root of the characteristic equation $\det(A - \lambda I_n) = 0$.

Example $A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

The characteristic polynomial was

$$P_A(\lambda) = -(3+\lambda)(\lambda-5)(\lambda-1) = -\lambda^3 + 3\lambda^2 + 13\lambda - 15.$$

Find an eigenvector for each eigenvalue.

Last time, we concluded that the three eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 5$ and $\lambda_3 = 1$. For $\lambda_2 = 5$, the matrix

$$A - 5I_3 = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -3 & 2 \\ 0 & 0 & -8 \end{bmatrix} \quad \text{with} \quad \mathrm{rref}(A - 5I_3) = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So a solution $\vec{x}$ to $(A - 5I_3)\vec{x} = \vec{0}_3$ would have entries, $x_1 = 3x_2$, x_2 is free, and $x_3 = 0$. The eigenvectors associated with $\lambda_2 = 5$ are

$$\vec{x} = t\langle 3, 1, 0\rangle, \quad t \neq 0.$$

An example eigenvector is $\vec{v}_2 = \langle 3, 1, 0\rangle$.

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$\lambda_1 = -3$ $\quad$ $(A - (-3)I_3)\vec{x} = \vec{0}_3$

$$A - (-3)I_3 = \begin{bmatrix} 7 & 3 & -1 \\ 1 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & \frac{-11}{32} \\ 0 & 1 & \frac{15}{32} \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{11}{32}x_3,$$
$$x_2 = \frac{-15}{32}x_3$$
$$x_3 \text{ - free}$$

$$\vec{x} = \left\langle \frac{11}{32}x_3, \frac{-15}{32}x_3, x_3 \right\rangle = x_3 \left\langle \frac{11}{32}, \frac{-15}{32}, 1 \right\rangle$$

An example, taking $x_3 = 32$, is

$$\vec{v}_1 = \langle 11, -15, 32 \rangle$$

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$\lambda_3 = 1$ $\quad (A - 1I_3)\vec{x} = \vec{0}_3$

$$A - 1I_3 = \begin{bmatrix} 3 & 3 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

x_2 - free

$x_3 = 0$

$\vec{x} = \langle -x_2, x_2, 0 \rangle = x_2 \langle -1, 1, 0 \rangle$

An example with $x_2 = 1$ is

$\vec{v}_3 = \langle -1, 1, 0 \rangle$

we found

$$\lambda_1 = -3 \qquad \vec{v}_1 = \langle 11, -15, 32 \rangle$$

$$\lambda_2 = 5 \qquad \vec{v}_2 = \langle 3, 1, 0 \rangle$$

$$\lambda_3 = 1, \qquad \vec{v}_3 = \langle -1, 1, 0 \rangle$$

Eigenspaces & Eigenbases

Definition

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The **eigenspace corresponding to the eigenvalue** λ_0 is the set

$$E_A(\lambda_0) = \{\vec{x} \in R^n \mid A\vec{x} = \lambda_0 \vec{x}\} = \mathcal{N}(A - \lambda_0 I_n).$$

An **eigenspace** is a null space, so it's a subspace of R^n. We can find a basis the way we regularly find the basis for a null space.

An **eigenspace** is all of the eigenvectors for a given eigenvalue with the zero vector thrown in to make a subspace.

Example

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$. Find the characteristic polynomials $P_A(\lambda)$ and $P_B(\lambda)$.

$$P_A(\lambda) = \det(A - \lambda I_3) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

$$= (2-\lambda)(2-\lambda)(4-\lambda)$$

$$= (2-\lambda)^2(4-\lambda)$$

$$P_B(\lambda) = \det(B - \lambda I_3) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

$$= (2-\lambda)^2(4-\lambda)$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \quad P_A(\lambda) = P_B(\lambda) = (2-\lambda)^2(4-\lambda)$$

$\lambda_1 = 2, \quad \lambda_2 = 4$

Find bases for the eigenspaces $E_A(2)$ and $E_B(2)$.

$E_A(2) = \mathcal{N}(A - 2I_3)$ we're solving $(A - 2I_3)\vec{x} = \vec{0}_3$

$$A - 2I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_1 - free
x_2 - free
$x_3 = 0$

$\vec{x} = \langle x_1, x_2, 0 \rangle = x_1 \langle 1, 0, 0 \rangle + x_2 \langle 0, 1, 0 \rangle$

A basis for $E_A(2)$ is $\{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle\}$.

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$E_B(2) = \mathcal{N}(B - 2I_3)$

solving $(B - 2I_3)\vec{x} = \vec{0}_3$

$$B - 2I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_1 - free

$x_2 = x_3 = 0$

$$\vec{x} = \langle x_1, 0, 0 \rangle = x_1 \langle 1, 0, 0 \rangle$$

A basis for $E_B(2)$ is $\{\langle 1, 0, 0 \rangle\}$.

Two Types of Multiplicities

Geometric Multiplicity

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The dimension of the eigenspace, $\mathrm{dim}(E_A(\lambda_0))$, corresponding to λ_0 is called the **geometric multiplicity** of λ_0.

To determine the geometric multiplicity, we have to find the dimension of the eigenspace—i.e., how many free variables are there in the equation

$$(A - \lambda_0 I_n)\vec{x} = \vec{0}_n.$$

Algebraic Multiplicity

Let A be an $n \times n$ matrix and λ_0 be an eigenvalue of A. The **algebraic multiplicity** of λ_0 is its multiplicity as the root of the characteristic equation $P_A(\lambda) = 0$. That is, if $(\lambda - \lambda_0)^k$ is a factor of $P_A(\lambda)$ and $(\lambda - \lambda_0)^{k+1}$ is not a factor of $P_A(\lambda)$, then the algebraic multiplicity of λ_0 is k.

If the characteristic polynomial was $(3-\lambda)^4(7-\lambda)^2(-2-\lambda)$, the eigenvalues with their algebraic multiplicities would be

$$\lambda = 3 \ \text{ algebraic multiplicity } \ 4,$$

$$\lambda = 7 \ \text{ algebraic multiplicity } \ 2,$$

$$\lambda = -2 \ \text{ algebraic multiplicity } \ 1.$$

The algebraic multiplicity is always greater than or equal to the geometric multiplicity.

Example

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P_A(\lambda) = (2-\lambda)^2(4-\lambda) \quad \text{and} \quad P_B(\lambda) = (2-\lambda)^2(4-\lambda).$$

Both have eigenvalue $\lambda = 2$ with **algebraic multiplicity** of two.

$$\underbrace{\{\langle 1,0,0\rangle, \langle 0,1,0\rangle\}}_{\text{basis for } E_A(2)} \qquad \underbrace{\{\langle 1,0,0\rangle\}}_{\text{basis for } E_B(2)}$$

$\lambda = 2$ has geometric multiplicity **two** as an eigenvalue of A and it has a geometric multiplicity **one** as an eigenvalue of B.

If a matrix has enough linearly independent eigenvectors, we may be able to build a basis for R^n out of eigenvectors. So the geometric multiplicity is of interest as is their linear independence.

Theorem

Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of eigenvectors of an $n \times n$ matrix corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

Note

If A is an $n \times n$ matrix with n distinct eigenvalues, then A has a set of n linearly independent eigenvectors.

Definition

Let A be an $n \times n$ matrix. If A has n linearly independent eigenvectors, $\vec{v}_1, \ldots, \vec{v}_n$ (combined across all eigenvalues), then the set $\mathcal{E}_A = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for R^n called an **eigenbasis** for A.

Suppose A is $n \times n$

- If A has n distinct eigenvalues, it is guaranteed to have an eigenbasis.
- If A has fewer than n distinct eigenvalues, then
  - it has an eigenbasis if the sum of all geometric multiplicities is n;
  - it doesn't have an eigenbasis if the sum of all geometric multiplicities is smaller than n.

Example

Find an eigenbasis for $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ or show that it is not possible.

Find the e. values.

$\det(A - \lambda I_2) = \det \begin{bmatrix} -2-\lambda & 8 \\ 1 & 5-\lambda \end{bmatrix}$

$= (-2-\lambda)(5-\lambda) - (1)(8)$

$= \lambda^2 - 3\lambda - 10 - 8$

$= \lambda^2 - 3\lambda - 18 = P_A(\lambda)$

$\lambda^2 - 3\lambda - 18 = 0 \implies (\lambda - 6)(\lambda + 3) = 0$

A has two eigenvalues, $\lambda_1 = 6$, $\lambda_2 = -3$.

$$A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$$

For λ_1, solve $(A - 6I_2)\vec{x} = \vec{0}_2$

$$A - 6I_2 = \begin{bmatrix} -8 & 8 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$x_1 = x_2$

x_2 - free

$\vec{x} = \langle x_2, x_2 \rangle = x_2 \langle 1, 1 \rangle$

A basis for $E_A(6)$ is $\{\langle 1, 1 \rangle\}$.

$A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ $\lambda_2 = -3$, $(A-(-3)I_2)\vec{x} = \vec{0}_2$.

$$A-(-3)I_2 = \begin{bmatrix} 1 & 8 \\ 1 & 8 \end{bmatrix} . \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 8 \\ 0 & 0 \end{bmatrix}$$

$x_1 = -8x_2$
x_2 - free

$$\vec{x} = (-8x_2, x_2) = x_2(-8, 1)$$

A basis for $E_A(-3)$ is $\{\langle -8, 1\rangle\}$.

An eigen basis for A is

$$\mathcal{E}_A = \{\langle 1, 1\rangle, \langle -8, 1\rangle\}.$$

Example: $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ $\qquad \lambda_1 = 6$, $\vec{v}_1 = \langle 1, 1 \rangle$ $\qquad \lambda_2 = -3$, $\vec{v}_2 = \langle -8, 1 \rangle$

1. Create a matrix C having the eigenvectors as its column vectors.
2. Find C^{-1}.
3. Find the product $C^{-1}AC$.

1. $C = \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$

2. Find C^{-1}: $[C \mid I_2] = \left[\begin{array}{cc|cc} 1 & -8 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$

$\xrightarrow{\text{rref}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{9} & \frac{8}{9} \\ 0 & 1 & -\frac{1}{9} & \frac{1}{9} \end{array}\right]$

$C^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix} \quad C^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}$$

3. $$C^{-1} A C = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 & 24 \\ 6 & -3 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 54 & 0 \\ 0 & -27 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$