December 2 Math 2306 sec. 51 Fall 2024

Review of Everything

Solve the initial value problem $t\dfrac{dy}{dt} - 3y = t^3\ln(t), \quad y(1) = 2.$

Standard form $\quad \dfrac{dy}{dt} - \dfrac{3}{t}y = t^2\ln t$

$P(t) = \dfrac{-3}{t}$, $\quad \mu = e^{\int P(t)\,dt} = e^{\int \frac{-3}{t}\,dt} = e^{-3\ln t}$

$= e^{\ln t^{-3}} = t^{-3}$

First order linear. Use an integrating factor.

$\dfrac{d}{dt}\left(t^{-3}y\right) = t^{-3}\left(t^2\ln t\right) = \dfrac{\ln t}{t}$

$$\int \frac{d}{dt}\left(t^{-3} y\right)' t = \int \frac{\ln t}{t}\, dt \qquad u = \ln t, \quad du = \frac{1}{t}\, dt$$

$$t^{-3} y = \int u\, du$$

$$t^{-3} y = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln t)^2 + C$$

$$y = \frac{\frac{1}{2} (\ln t)^2 + C}{t^{-3}} = \frac{1}{2} t^3 (\ln t)^2 + C t^3$$

Apply $y(1) = 2$

$$y(1) = \frac{1}{2} (1)^3 (\ln 1)^2 + C(1)^3$$

$$= 0 + C = 2 \Rightarrow C = 2$$

$$y = \frac{1}{2} t^3 (\ln t)^2 + 2t^3$$

Find the general solution.

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Reduction of
order to finish
out yc, then
v.o.p. to find
yp.
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$$x^2y'' - 4xy' + 4y = 18x$$

(Hint: $y_1 = x$ solves the homogeneous equation.)

$y = y_c + y_p$, $y_c = c_1 y_1 + c_2 y_2$

The homogeneous ODE is $x^2 y'' - 4xy' + 4y = 0$

Use reduction of order to find y_2.

$y_2 = u y_1$ $\quad u = \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$

Standard form $\quad y'' - \frac{4}{x} y' + \frac{4}{x^2} y = 0 \quad P(x) = \frac{-4}{x}$

$e^{-\int P(x)dx} = e^{\int \frac{4}{x} dx} = e^{4 \ln x} = x^4$

$u = \int \frac{x^4}{x^2}\,dx = \int x^2\,dx = \frac{x^3}{3}$, $y_2 = u y_1 = \frac{x^3}{3}\cdot x$

$y_2 = \frac{1}{3}x^4$. $y_c = c_1 y_1 + c_2 y_2 = c_1 x + c_2\left(\frac{1}{3}x^4\right)$

we can take $y_2 = x^4$, $y_c = c_1 x + c_2 x^4$

$$x^2 y'' - 4xy' + 4y = 18x$$

$$y'' - \frac{4}{x}y' + \frac{4}{x^2}y = \frac{18}{x}$$

V.O.P. $y_p = u_1 y_1 + u_2 y_2$ $y_1 = x$, $y_2 = x^4$

$u_1 = \int \frac{-y_2 g}{W}\,dx$, $u_2 = \int \frac{y_1 g}{W}\,dx$ $g(x) = \frac{18}{x}$

$W = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix} = 4x^4 - x^4 = 3x^4$

$$u_1 = \int \frac{-x^4\left(\frac{18}{x}\right)}{3x^4}\,dx = -6\int \frac{1}{x}\,dx = -6\ln|x|$$

$$u_2 = \int \frac{x\left(\frac{18}{x}\right)}{3x^4}\,dx = 6\int x^{-4}\,dx = \frac{6}{-3}x^{-3} = -2x^{-3}$$

$$y_p = \left(-6\ln|x|\right)x + \left(-2x^{-3}\right)x^4 = -6x\ln|x| - 2x$$

$$y = y_c + y_p = c_1x + c_2x^4 - 6x\ln|x| - 2x$$

$$\boxed{y = k_1x + k_2x^4 - 6x\ln|x|}$$

$k_1 = c_1 - 2, \quad k_2 = c_2$

Find the function f that satisfies the equation

$$f(t) = \int_0^t f(\tau)(t-\tau)\,d\tau + 1.$$

Let $\mathcal{L}\{f(t)\} = F(s)$

$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$ $\quad g(t) = t$

$\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$ $\quad G(s) = \frac{1!}{s^2}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{(f*g)(t) + 1\}$

$F(s) = F(s)\left(\frac{1!}{s^2}\right) + \frac{1}{s}$

$F(s) = \frac{1}{s^2}F(s) + \frac{1}{s}$

$$F(s)\left(1-\frac{1}{s^2}\right)=\frac{1}{s}$$

$$F(s)\left(\frac{s^2-1}{s^2}\right)=\frac{1}{s} \Rightarrow F(s)=\frac{1}{s}\left(\frac{s^2}{s^2-1}\right)$$

$$F(s)=\frac{s}{s^2-1}=\frac{s}{(s-1)(s+1)}$$

$$\frac{s}{(s-1)(s+1)}=\frac{A}{s-1}+\frac{B}{s+1} \Rightarrow s=A(s+1)+B(s-1)$$

$s=1 \quad 1=2A, \quad A=\frac{1}{2}$

$s=-1 \quad -1=-2B \quad B=\frac{1}{2}$

$$F(s)=\frac{\frac{1}{2}}{s-1}+\frac{\frac{1}{2}}{s+1}$$

$$f(t)=\mathcal{L}^{-1}\{F(s)\}$$

$$\boxed{f(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}}$$

Solve the initial value problem

It's Bernoulli!

$$xy' + y = x^4y^4, \quad y(1) = 1.$$

$u = y^{1-n}$, u solves $\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$

Standard form, $y' + \frac{1}{x}y = x^3y^4$

$n = 4$, $P(x) = \frac{1}{x}$, $f(x) = x^3$; $1-n = 1-4 = -3$

$u = y^{-3}$, u solves

$$\frac{du}{dx} + (-3)\left(\frac{1}{x}\right)u = -3x^3$$

$u' - \frac{3}{x}u = -3x^3$, $P_1(x) = \frac{-3}{x}$,

$$\mu = e^{\int P_1(x)dx} = e^{\int -3/x \, dx} = e^{-3\ln x} = x^{-3}$$

$$\frac{d}{dx}\left(x^{-3}u\right) = -3x^{3}\left(x^{-3}\right) = -3$$

$$\int \frac{d}{dx}\left(x^{-3}u\right) dx = \int -3dx = -3x + C$$

$$x^{-3}u = -3x + C \Rightarrow u = -3x^{4} + Cx^{3}$$

$$u = y^{-3} \Rightarrow y = u^{-1/3} = \left(-3x^{4} + Cx^{3}\right)^{-1/3}$$

$$y(1) = \left(-3(1)^{4} + C(1)^{3}\right)^{-1/3} = (C-3)^{-1/3} = 1$$

$$1 = \frac{1}{\sqrt[3]{C-3}} \Rightarrow$$

$$\sqrt[3]{c-3} = \frac{1}{1} = 1$$

$$c-3 = 1^3 = 1 \Rightarrow c = 4$$

$$\boxed{y = \left(4x^3 - 3x^4\right)^{-1/3}}$$