December 2 Math 2306 sec. 51 Fall 2024

Review of Everything

Solve the initial value problem $t\dfrac{dy}{dt} - 3y = t^3\ln(t), \quad y(1) = 2.$

Standard form $\quad \dfrac{dy}{dt} - \dfrac{3}{t}y = t^2 \ln t$

$P(t) = \dfrac{-3}{t}$ $\qquad \mu = e^{\int P(t)\,dt} = e^{\int \frac{-3}{t}dt} = e^{-3\ln t} = e^{\ln t^{-3}}$

$= t^{-3}$

$\dfrac{d}{dt}\left(t^{-3}y\right) = t^{-3}\left(t^2 \ln t\right) = \dfrac{\ln t}{t}$

$$\int \frac{d}{dt}\left(t^{-3}y\right)dt = \int \frac{\ln t}{t}\,dt = \int \ln t\left(\frac{1}{t}\right)dt$$

$u = \ln t \quad , \quad du = \frac{1}{t}\,dt$

$$t^{-3}y = \int u\,du$$

$$= \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}(\ln t)^2 + C$$

$$y = \frac{\frac{1}{2}(\ln t)^2 + C}{t^{-3}} = \frac{1}{2}t^3(\ln t)^2 + Ct^3$$

Apply $y(1) = 2$

$$y(1) = \frac{1}{2}(1)^3(\ln 1)^2 + C(1)^3 = 2$$

$$C = 2$$

$$y = \frac{1}{2} t^3 (\ln t)^2 + 2 t^3$$

Find the general solution.

$$x^2 y'' - 4xy' + 4y = 18x$$

(Hint: $y_1 = x$ solves the homogeneous equation.)

General solution is $y = y_c + y_p$ where

$y_c = c_1 y_1 + c_2 y_2$

Use reduction of order to find y_2.

$y_2 = u y_1$, $u = \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$

Standard form: $y'' - \frac{4}{x} y' + \frac{4}{x^2} y = \frac{18}{x}$

$P(x) = \frac{-4}{x}$, $\quad e^{-\int P(x)\,dx} = e^{\int \frac{4}{x}\,dx} = e^{4\ln|x|} = x^4$

$$u = \int \frac{x^4}{x^2}\,dx = \int x^2\,dx = \frac{1}{3}x^3$$

$$y_2 = \frac{1}{3}x^3 \cdot x = \frac{1}{3}x^4$$

$$y_c = c_1 y_1 + c_2 y_2$$

we can take. $y_1 = x$, $y_2 = x^4$

$$y'' - \frac{4}{x}y' + \frac{4}{x^2}y = \frac{18}{x}$$

Now, we find y_p. Use V.O.P.

$$y_p = u_1 y_1 + u_2 y_2 , \quad u_1 = \int \frac{-y_2 g}{W}\,dx , \quad u_2 = \int \frac{y_1 g}{W}\,dx$$

$$g(x) = \frac{18}{x} , \quad W = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix} = 4x^4 - x^4 = 3x^4$$

$$= x^4(4-1) = 3x^4$$

$$u_1 = \int \frac{-x^4\left(\frac{18}{x}\right)}{3x^4}\,dx = -6\int \frac{1}{x}\,dx = -6\ln|x|$$

$$u_2 = \int \frac{x\left(\frac{18}{x}\right)}{3x^4}\,dx = 6\int x^{-4}\,dx = \frac{6}{-3}x^{-3} = -2x^{-3}$$

$$y_p = (-6\ln|x|)\,x + (-2x^{-3})\,x^4$$

$$= -6x\ln|x| - 2x$$

$$y = y_c + y_p$$

$$y = c_1 x + c_2 x^4 - 6x\ln|x| - 2x$$

we could write

$$\boxed{y = c_1 x + c_2 x^4 - 6x\ln|x|}$$

Find the function f that satisfies the equation

$$f(t) = \int_0^t f(\tau)(t-\tau)\,d\tau + 1.$$

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$\mathcal{L}\{(f*g)(t)\} = F(s)\,G(s)$$

If $g(t-\tau) = t-\tau$ then $g(t) = t$

$$\mathcal{L}\{t\} = \frac{1!}{s^2}$$

Letting $F(s) = \mathcal{L}\{f(t)\}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(f*g)(t) + 1\}$$

$$F(s) = F(s)\left(\frac{1}{s^2}\right) + \frac{1}{s}$$

Isolate $F(s)$

$$F(s)\left(1 - \frac{1}{s^2}\right) = \frac{1}{s}$$

$$F(s)\left(\frac{s^2-1}{s^2}\right) = \frac{1}{s}$$

$$F(s) = \frac{1}{s}\left(\frac{s^2}{s^2-1}\right) = \frac{s}{s^2-1}$$

$$\frac{s}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \Rightarrow s = A(s+1) + B(s-1)$$

$s = 1 \quad 1 = 2A \quad A = \frac{1}{2}$

$s = -1 \quad -1 = -2B \quad B = \frac{1}{2}$

$$F(s) = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$\boxed{f(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}}$$

$y' + p(x) y = f(x) y^n$

Solve the initial value problem

$$xy' + y = x^4 y^4, \quad y(1) = 1.$$

$u = y^{1-n}$, u solves $\frac{du}{dx} + (1-n) p(x) u = (1-n) f(x)$

$n = 4$, $1-n = 1-4 = -3$

$$y' + \frac{1}{x} y = x^3 y$$

$$p(x) = \frac{1}{x}, \quad f(x) = x^3$$

$$u = y^{1-4} = y^{-3} \iff y = u^{-1/3}$$

Okay, so $u = y^{-3}$ and u solves

$$\frac{du}{dx} + (-3)\left(\frac{1}{x}\right) u = -3x^3$$

$$u' - \frac{3}{x} u = -3x^3$$ 1st order linear

$$P_1(x) = \frac{-3}{x}, \quad \mu = e^{\int P_1(x)dx} = e^{\int \frac{-3}{x} dx} = e^{-3\ln x} = x^{-3}$$

$$\frac{d}{dx}\left(x^{-3} u\right) = x^{-3}(-3x^3) = -3$$

$$x^{-3} u = \int -3\,dx = -3x + C$$

$$u = \frac{-3x + C}{x^{-3}} = -3x^4 + Cx^3$$

Since $y = u^{-1/3}$, $y = \left(-3x^4 + cx^3\right)^{-1/3}$.

Now, apply $y(1) = 1$.

$$y(1) = \left(-3(1)^4 + c(1)^3\right)^{-1/3} = \frac{1}{\sqrt[3]{c-3}} = 1$$

$$\sqrt[3]{c-3} = \frac{1}{1} = 1$$

$$c - 3 = 1^3 = 1$$

$$c = 4$$

Finally $\boxed{y = \dfrac{1}{\sqrt[3]{4x^3 - 3x^4}}}$