December 3 Math 3260 sec. 51 Fall 2025

6.2 Eigenvalues & Eigenvectors

We had defined the **eigenvalues** and **eigenvectors** of a square matrix A as the numbers λ and nonzero vectors $\vec{x}$ that satisfy the equation

$$A\vec{x} = \lambda\vec{x}. \tag{1}$$

The eigenvalues are the solutions of the **characteristic equation**

$$P_A(\lambda) = \det(A - \lambda I_n) = 0$$

For a given eigenvalue λ of a matrix A, the **eigenspace** corresponding to λ is the subspace of R^n

$$E_A(\lambda) = \{\vec{x} \in R^n \mid A\vec{x} = \lambda\vec{x}\} = \mathcal{N}(A - \lambda I_n)$$

$A\vec{x} = \lambda\vec{x}$

Recall: An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

- $\det(A) = 0 \quad \Longrightarrow \quad A$ is **not** invertible,
- $\det(A) \neq 0 \quad \Longrightarrow \quad A$ **is** invertible.

Theorem

An $n \times n$ matrix A is invertible if and only if zero is **not** and eigenvalue of A.

Question: What is the connection, if any, between $\det(A - \lambda I_n)$ and $\det(A)$ if $\lambda = 0$?

$$\det(A - \lambda I_n) \underset{\lambda=0}{=} \det(A - 0I_n) = \det(A)$$

$A\vec{x} = \lambda\vec{x}$

Each eigenvalue λ_i for a matrix A has two kinds of multiplicities:

- **Algebraic**: as a root of a polynomial—the power k on the factor $(\lambda - \lambda_i)^k$, and
- **Geometric**: dimension of the eigenspace—the number of free variables for $(A - \lambda_i I_n)\vec{x} = \vec{0}_n$.

If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are eigenvectors corresponding to **distinct** eigenvalues $\lambda_1, \ldots, \lambda_k$, then the set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is guaranteed to be **linearly independent**.

Definition

Let A be an $n \times n$ matrix. If A has n linearly independent eigenvectors, $\vec{v}_1, \ldots, \vec{v}_n$ (combined across all eigenvalues), then the set $\mathcal{E}_A = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for R^n called an **eigenbasis** for A.

Suppose A is $n \times n$

- If A has n distinct eigenvalues, it is guaranteed to have an eigenbasis.
- If A has fewer than n distinct eigenvalues, then
  - it has an eigenbasis if the sum of all geometric multiplicities is n;
  - it doesn't have an eigenbasis if the sum of all geometric multiplicities is smaller than n.

Example: $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ $\quad \lambda_1 = 6,\ \vec{v}_1 = \langle 1, 1 \rangle \quad \lambda_2 = -3,\ \vec{v}_2 = \langle -8, 1 \rangle$

1. Create a matrix C having the eigenvectors as its column vectors.
2. Find C^{-1}.
3. Find the product $C^{-1}AC$.

1. $C = \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$

2. $[C \mid I_2] = \left[\begin{array}{cc|cc} 1 & -8 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{9} & \frac{8}{9} \\ 0 & 1 & -\frac{1}{9} & \frac{1}{9} \end{array}\right]$

$$C^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} \\ -\frac{1}{9} & \frac{1}{9} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}$$

Example: $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ $\lambda_1 = 6$ $\vec{v}_1 = \langle 1, 1 \rangle$ $\lambda_2 = -3$ $\vec{v}_2 = \langle -8, 1 \rangle$

$$C = \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix} \qquad C^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}$$

$$C^{-1}AC = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 & 24 \\ 6 & -3 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 54 & 0 \\ 0 & -27 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

6.3 Diagonalization

Definition

An $n \times n$ matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix. That is, A is diagonalizable if there exists a diagonal matrix D and an invertible matrix C such that

$$D = C^{-1}AC. \qquad A = CDC^{-1}$$

The previous example suggests that diagonalizability is related to making a matrix out of eigenvectors. This turns out to be true, but to get an $n \times n$ matrix that is actually invertible, we need n linearly independent vectors. This is where having an eigenbasis comes in.

Facts About Similar Matrices

Theorem

If A and B are similar matrices, the $\det(A) = \det(B)$.

Theorem

If A and B are similar matrices, then A and B have the same eigenvalues, each with the same algebraic and geometric multiplicities.

If A and B are similar, so they share an eigenvalue λ, the eigenvectors corresponding to λ are **generally different**.

$B = C^{-1}AC$

Show that $\det(B) = \det(A)$ and $P_B(\lambda) = P_A(\lambda)$.

$$\det(B) = \det(C^{-1}AC)$$
$$= \det(C^{-1})\det(A)\det(C)$$
$$= \det(A)\det(C^{-1})\det(C)$$
$$= \det(A)\det(C^{-1}C)$$
$$= \det(A)\det(I_n)$$
$$= \det(A)\cdot 1$$
$$= \det(A).$$

since det's are scalar

$$B = C^{-1}AC$$

$$P_B(\lambda) = \det(B - \lambda I_n)$$

$$B - \lambda I_n = C^{-1}AC - \lambda I_n \quad , \quad I_n = C^{-1} I_n C$$

$$= C^{-1}AC - \lambda C^{-1} I_n C$$

$$= C^{-1}(AC - \lambda I_n C)$$

$$= C^{-1}(A - \lambda I_n) C$$

$$P_B(\lambda) = \det(B - \lambda I_n)$$

$$= \det(C^{-1}(A - \lambda I_n) C)$$

$$= \det(C^{-1}) \det(A - \lambda I_n) \det(C)$$

$$= \det(A - \lambda I_n) = P_A(\lambda)$$

Theorem

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

Moreover, if A is diagonalizable, then there exists a diagonal matrix D such that $D = C^{-1}AC$ where the columns of the invertible matrix C are the vectors in an eigenbasis, $\mathcal{E}_A$, for the matrix A, and the diagonal entries of the matrix D are the eigenvalues of A.

Big Idea 1: If A has n distinct eigenvalues, then it is guaranteed to be diagonalizable. If it has less than n distinct eigenvalues, it may or may not be diagonalizable.

A is diagonalizable if the sum of the geometric multiplicities is n.

Theorem

Moreover, if A is diagonalizable, then there exists a diagonal matrix D such that $D = C^{-1}AC$ where the columns of the invertible matrix C are the vectors in an eigenbasis, $\mathcal{E}_A$, for the matrix A, and the diagonal entries of the matrix D are the eigenvalues of A.

Big Idea 2: If A is diagonalizable, then

$$C = \underbrace{\begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix}}_{\text{columns are e. vecs}} \quad \text{and} \quad D = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{entries are e. vals}}$$

Example

Let $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. The characteristic polynomial $P_A(\lambda) = (1-\lambda)(2+\lambda)^2$. Determine whether A is diagonalizable.

Find e. values. $P_A(\lambda) = 0 \Rightarrow (1-\lambda)(2+\lambda)^2 = 0$

$\lambda_1 = 1$ and $\lambda_2 = -2$

Find eigenvectors for $\lambda_2 = -2$

solve $(A - (-2)I_3)\vec{x} = \vec{0}_3$

$$A - (-2)I_3 = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2,\quad x_2 - \text{free} \quad x_3 = 0$

Geometric mult for λ_2 is 1.

A is not diagonalizable.

A will have one linearly independent eigenvector for the other eigenvalue (1). But since A only has one linearly independent eigenvector for -2, we only get two linearly independent eigenvectors in total. Since A is 3x3, we'd need three to build an invertible matrix. Hence A is not diagonalizable.