December 3 Math 3260 sec. 53 Fall 2025

6.2 Eigenvalues & Eigenvectors

We had defined the **eigenvalues** and **eigenvectors** of a square matrix A as the numbers λ and nonzero vectors $\vec{x}$ that satisfy the equation

$$A\vec{x} = \lambda\vec{x}. \tag{1}$$

The eigenvalues are the solutions of the **characteristic equation**

$$P_A(\lambda) = \det(A - \lambda I_n) = 0$$

For a given eigenvalue λ of a matrix A, the **eigenspace** corresponding to λ is the subspace of R^n

$$E_A(\lambda) = \{\vec{x} \in R^n \mid A\vec{x} = \lambda\vec{x}\} = \mathcal{N}(A - \lambda I_n)$$

$A\vec{x} = \lambda\vec{x}$

Recall: An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

- $\det(A) = 0 \implies A$ is **not** invertible,
- $det(A) \neq 0 \implies A$ **is** invertible.

Theorem

An $n \times n$ matrix A is invertible if and only if zero is **not** and eigenvalue of A.

Question: What is the connection, if any, between $\det(A - \lambda I_n)$ and $\det(A)$ if $\lambda = 0$?

$\det(A - \lambda I_n) = \det(A - 0 I_n) = \det(A)$

$\lambda = 0$

$A\vec{x} = \lambda\vec{x}$

Each eigenvalue λ_i for a matrix A has two kinds of multiplicities:

- **Algebraic**: as a root of a polynomial—the power k on the factor $(\lambda - \lambda_i)^k$, and
- **Geometric**: dimension of the eigenspace—the number of free variables for $(A - \lambda_i I_n)\vec{x} = \vec{0}_n$.

If $\{\vec{v}_1, \dots, \vec{v}_k\}$ are eigenvectors corresponding to **distinct** eigenvalues $\lambda_1, \dots, \lambda_k$, then the set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is guaranteed to be **linearly independent**.

Definition

Let A be an $n \times n$ matrix. If A has n linearly independent eigenvectors, $\vec{v}_1, \ldots, \vec{v}_n$ (combined across all eigenvalues), then the set $\mathcal{E}_A = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for R^n called an **eigenbasis** for A.

Suppose A is $n \times n$

- If A has n distinct eigenvalues, it is guaranteed to have an eigenbasis.
- If A has fewer than n distinct eigenvalues, then
  - it has an eigenbasis if the sum of all geometric multiplicities is n;
  - it doesn't have an eigenbasis if the sum of all geometric multiplicities is smaller than n.

Example: $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ $\quad \begin{array}{c} \lambda_1 = 6 \\ \vec{v}_1 = \langle 1, 1 \rangle \end{array} \quad \begin{array}{c} \lambda_2 = -3 \\ \vec{v}_2 = \langle -8, 1 \rangle \end{array}$

1. Create a matrix C having the eigenvectors as its column vectors.
2. Find C^{-1}.
3. Find the product $C^{-1}AC$.

We set up $C = \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix}$. The inverse turned out to be $C^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix}$. Note that

$$\begin{aligned} AC &= \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 24 \\ 6 & -3 \end{bmatrix} \\ &= \begin{bmatrix} | & | \\ \lambda_1 \vec{v}_1 & \lambda_2 \vec{v}_2 \\ | & | \end{bmatrix} \end{aligned}$$

Example: $A = \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix}$ $\quad \begin{matrix} \lambda_1 = 6 \\ \vec{v}_1 = \langle 1, 1 \rangle \end{matrix} \quad \begin{matrix} \lambda_2 = -3 \\ \vec{v}_2 = \langle -8, 1 \rangle \end{matrix}$

So the final result ended up

$$\begin{aligned} C^{-1}AC &= \frac{1}{9} \begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 8 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 1 & 8 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 24 \\ 6 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 \\ 0 & -3 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \end{aligned}$$

So $C^{-1}AC$ is a **diagonal matrix** with the eigenvalues on the main diagonal. The order they appear in matches the order we chose for the eigenvectors when creating $C = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$.

6.3 Diagonalization

Definition

An $n \times n$ matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix. That is, A is diagonalizable if there exists a diagonal matrix D and an invertible matrix C such that

$$D = C^{-1}AC.$$

$A = CDC^{-1}$

The previous example suggests that diagonalizability is related to making a matrix out of eigenvectors. This turns out to be true, but to get an $n \times n$ matrix that is actually invertible, we need n linearly independent vectors. This is where having an eigenbasis comes in.

Facts About Similar Matrices

Theorem

If A and B are similar matrices, the $\det(A) = \det(B)$.

Theorem

If A and B are similar matrices, then A and B have the same eigenvalues, each with the same algebraic and geometric multiplicities.

If A and B are similar, so they share an eigenvalue λ, the eigenvectors corresponding to λ are **generally different**.

$B = C^{-1}AC$

Show that $\det(B) = \det(A)$ and $P_B(\lambda) = P_A(\lambda)$.

$$\begin{aligned}
\det(B) &= \det(C^{-1}AC) \\
&= \det(C^{-1})\det(A)\det(C) \\
&= \det(A)\det(C^{-1})\det(C) \\
&= \det(A)\det(C^{-1}C) \\
&= \det(A)\det(I_n) \\
&= \det(A)\cdot 1 \\
&= \det(A)
\end{aligned}$$

these are scalars

$B = C^{-1}AC$

$P_B(\lambda) = \det(B - \lambda I_n)$

$$B - \lambda I_n = C^{-1} A C - \lambda I_n \quad , \quad I_n = C^{-1} I_n C$$

$$= C^{-1} A C - \lambda C^{-1} I_n C$$

$$= C^{-1} (A C - \lambda I_n C)$$

$$= C^{-1} (A - \lambda I_n) C$$

$$P_B(\lambda) = \det(B - \lambda I_n)$$

$$= \det\left(C^{-1}(A - \lambda I_n) C\right)$$

$$= \det(A - \lambda I_n)$$

$$= P_A(\lambda).$$

Theorem

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

Moreover, if A is diagonalizable, then there exists a diagonal matrix D such that $D = C^{-1}AC$ where the columns of the invertible matrix C are the vectors in an eigenbasis, $\mathcal{E}_A$, for the matrix A, and the diagonal entries of the matrix D are the eigenvalues of A.

Big Idea 1: If A has n distinct eigenvalues, then it is guaranteed to be diagonalizable. If it has less than n distinct eigenvalues, it may or may not be diagonalizable.

A is diagonalizable if the sum of the geometric multiplicities is n.

Theorem

Moreover, if A is diagonalizable, then there exists a diagonal matrix D such that $D = C^{-1}AC$ where the columns of the invertible matrix C are the vectors in an eigenbasis, $\mathcal{E}_A$, for the matrix A, and the diagonal entries of the matrix D are the eigenvalues of A.

Big Idea 2: If A is diagonalizable, then

$$C = \underbrace{\begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix}}_{\text{columns are e. vecs}} \quad \text{and} \quad D = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\text{entries are e. vals}}$$

Example

Let $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. The characteristic polynomial $P_A(\lambda) = (1-\lambda)(2+\lambda)^2$. Determine whether A is diagonalizable.

Find the e. vals. $P_A(\lambda) = 0$

$(1-\lambda)(2+\lambda)^2 = 0 \Rightarrow \lambda_1 = 1$ and $\lambda_2 = -2$

Look for e. vectors.

For $\lambda_1 = 1$

$A - 1I_3 = \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$(A - 1I_3)\vec{x} = \vec{0}_3$

$x_1 = x_3$
$x_2 = -x_3$
x_3 - free

$\vec{x} = (x_3, -x_3, x_3) = x_3\langle 1, -1, 1\rangle$

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

For $\lambda_2 = -2$

$$A - (-2)I_3 = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$(A+2I_3)\vec{x} = \vec{0}_3$

Only one free variable.

we only get two lin. independent eigenvectors. There is no eigen basis.

A is not diagonalizable.

A 3x3 matrix would need three lin. independent eigenvectors. But we only got two. This wasn't really obvious until we actually found the geometric multiplicity of -2.

Example

Diagonalize the matrix $A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$ if possible.

Find e.vals.

$$\det(A - \lambda I_2) = \det \begin{bmatrix} -4-\lambda & 3 \\ -6 & 5-\lambda \end{bmatrix}$$

$$= (-4-\lambda)(5-\lambda) - (-6)(3)$$

$$= \lambda^2 - \lambda - 20 + 18$$

$$= \lambda^2 - \lambda - 2 \qquad = P_A(\lambda)$$

$$\lambda^2 - \lambda - 2 = 0 \implies (\lambda - 2)(\lambda + 1) = 0$$

$\lambda_1 = 2$ and $\lambda_2 = -1$

$$A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$$ Find eigen vectors

$\lambda_1 = 2$ $\quad A - 2I_2 = \begin{bmatrix} -6 & 3 \\ -6 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$(A - \lambda I_2)\vec{x} = \vec{0}_2$

$x_1 = \frac{1}{2}x_2.$

x_2 - free

$\vec{x} = \langle \frac{1}{2}x_2, x_2 \rangle = x_2 \langle \frac{1}{2}, 1 \rangle$

using $x_2 = 2$, an eigenvector is

$\vec{v}_1 = \langle 1, 2 \rangle.$

$\lambda_2 = -1$, $\quad \vec{v}_2 = \langle 1, 1 \rangle$

$C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

Here's the work for $\lambda_2 = -1$

$$A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$$

$$A - (-1)I_2 = \begin{bmatrix} -3 & 3 \\ -6 & 6 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

so $(A + I_2)\vec{x} = \vec{0}_2 \Rightarrow$ $x_1 = x_2$
x_2 - free

$$\vec{x} = \langle x_2, x_2 \rangle = x_2 \langle 1, 1 \rangle$$

we know that the columns of C are the eigenvectors. So

$$C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

or $C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

And we know that $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, so

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

or $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ for the other C choice

If we actually multiply

$C^{-1}AC$ we will get $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.