Exam 1 Math 2306 sec. 52

Fall 2021

Name: (4 pts) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
Total (+4)	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.
Show all of your work on the paper provided to receive full credit.

1. Classify each first order differential equation. That is, identify each equation as being Linear, Separable, or Bernoulli. (If an equation is of more than one type, you will receive full credit for correctly stating either type.)

(a) $\theta r = \dfrac{dr}{d\theta} + \theta^2$ Linear

$\dfrac{dr}{d\theta} - \theta r = -\theta^2$

(b) $\dfrac{dx}{dt} + x^2\cos(t) = 0$ Bernoulli and Separable

$x' = -x^2 \cos t$

(c) $x\dfrac{dy}{dx} + xy = \dfrac{1}{y}$ Bernoulli

(d) $\dfrac{dx}{dt} = e^{ax+bt}$, where a and b are positive real numbers. Separable

$x' = e^{ax} \cdot e^{bt}$

2. Solve the initial value problem. Your solution can be implicit or explicit, your choice.

$$\frac{dx}{dt} = \frac{e^x}{\sqrt{t}}, \quad x(1) = 0$$

The ODE is separable.

$$e^{-x}\frac{dx}{dt} = t^{-1/2}$$

$$\int e^{-x}\,dx = \int t^{-1/2}\,dt \Rightarrow -e^{-x} = 2t^{1/2} + C$$

$$e^{-x} = k - 2\sqrt{t} \qquad k = -C$$

Apply the I.C. $e^0 = k - 2\sqrt{1} \Rightarrow 1 = k - 2 \Rightarrow k = 3$

$$\boxed{\text{The solution is given by } e^{-x} = 3 - 2\sqrt{t}.}$$

Explicitly, this gives $x = -\ln(3 - 2\sqrt{t})$

3. Find all possible values of the constant m such that the function $y = x^m$ solves the differential equation

$$x^2y'' + 3xy' - y = 0.$$

$y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

Substitute $x^2\left[m(m-1)x^{m-2}\right] + 3x\left[mx^{m-1}\right] - x^m = 0$

$$x^m(m^2 - m + 3m - 1) = 0$$

This will hold if $m^2 + 2m - 1 = 0$.

Solving for m $\quad m^2 + 2m + 1 = 2$

$$(m+1)^2 = 2$$

$$m + 1 = \pm\sqrt{2}$$

$$\boxed{\text{There are two } m\text{-values} \quad m = -1 \pm \sqrt{2}.}$$

4. Consider the Bernoulli equation $\boxed{x\dfrac{dy}{dx} - y = \dfrac{x}{y^2}}$. Answer the following questions. (Note that you are not being asked to solve this ODE.)

(a) What[1] is the value of n?

$\boxed{n = -2}$

$\frac{x}{y^2} = xy^{-2}$

(b) What is the new variable u in terms of y?

$\boxed{u = y^3}$

$1-n = 1+2 = 3$

(c) What is y in terms of u?

$y = u^{1/3}$ a.k.a $\boxed{y = \sqrt[3]{u}}$

(d) What is the linear equation (in standard form) satisfied by the new variable u?

$y' - \frac{1}{x}y = \frac{1}{y^2}$

$P(x) = \frac{-1}{x}$ $\quad 1-n = 3$

$f(x) = 1$

$$\boxed{\frac{du}{dx} - \frac{3}{x}u = 3}$$

5. Provide a short answer for each question. Your answers should be clear and contain correct terminology or symbols.

(a) Consider the problem $y'' + 9y = 0$ subject to $y(0) = 1$, $y\left(\frac{\pi}{2}\right) = 2$. Is this an **initial value problem**? (Why/why not?)

No. The conditions for an IVP are given at one input. These conditions are @ two, 0 and $\frac{\pi}{2}$.

(b) Jack solves the linear equation $xy' - y = \dfrac{x^3}{x^2+1}$ and finds $y = \frac{x}{2}\ln(x^2+1) + Cx$. Identify the **complementary solution** and state how you know that you've identified the correct part.

The complementary solution solves $xy' - y = 0$ and has the parameter C in it.

$y_c = Cx$.

[1] I'm using the notational convention from this class when refering to n.

6. Solve the initial value problem using any applicable technique. Provide an explicit solution.

$$x\frac{dy}{dx} - 2y = x^2, \quad y(1) = -1$$

This is linear.

In standard form $y' - \frac{2}{x} y = x$

$P(x) = \frac{-2}{x}$. so $\mu = e^{\int \frac{-2}{x} dx} = e^{-2\ln x} = x^{-2}$

$$\frac{d}{dx}\left(x^{-2} y\right) = x^{-2}(x) = x^{-1}$$

$$x^{-2} y = \int x^{-1}\, dx = \ln|x| + C$$

$$\Rightarrow \quad y = \frac{\ln|x| + C}{x^{-2}} = x^2\left(\ln|x| + C\right)$$

The general solution is

$$y = x^2\left(\ln|x| + C\right).$$

Applying the IC

$$y(1) = 1^2\left(\ln|1| + C\right) = -1$$

$$0 + C = -1 \quad \Rightarrow \quad C = -1$$

The solution to the IVP

$$y = x^2\left(\ln|x| - 1\right)$$