Exam 2 Review Question Solutions: Math 2335 (Ritter)

Sections Covered: 4.1, 4.2, 4.3, 4.5-4.6, 5.1, 5.2, 5.3

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(1) Use a linear interpolation of $f(x) = \sin x$ to approximate $\sin(0.35)$ using the points on the graph $(0.3, 0.2955202)$ and $(0.4, 0.3894183)$. $\sin(0.35) \approx P_1(0.35) = 0.34246925$

(2) Find the unique polynomial of degree ≤ 2 that passes through the points $(1, 1)$, $(2, 4)$ and $(3, -1)$. $P_2(x) = -4x^2 + 15x - 10$

(3) Given $f(x) = e^{-x}$ and the x-values $x_0 = 0, x_1 = 0.1, x_2 = 0.2$, compute the divided differences

$$f[x_0, x_1], \quad f[x_1, x_2], \quad \text{and} \quad f[x_0, x_1, x_2].$$

Use the result to write the quadratic interpolation of $f(x)$ through the points $(0, 1)$, $(0.1, e^{0.1})$, $(0.2, e^{0.2})$. Use a TI89 or similar calculator in float 7 mode.

$$f[x_0, x_1] = 10(e^{0.1} - 1) \doteq 1.0517091, \quad f[x_1, x_2] = 10(e^{0.2} - e^{0.1}) \doteq 1.1623184,$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \doteq 0.5530461$$

$$\begin{aligned} P_2(x) &= f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= 1 + 1.0517091x + 0.5530461x(x - 0.1) \\ &= 0.553046x^2 + 0.996404x + 1 \end{aligned}$$

(4) Consider using a 3^{rd} order interpolating polynomial $P_3(t)$ to approximate the function $f(t) = \tan^{-1} t$ on the interval $-1 \leq t \leq 1$. Find the x-values x_0, x_1, x_2, x_3 that will minimize the error. We want the Chebyshev nodes, roots of T_4. These are

$$x_j = \cos\left(\frac{(2j + 1)\pi}{8}\right), \quad j = 0, \ldots, 3$$

$$x_0 \doteq 0.9238795, \quad x_1 \doteq 0.3826834, \quad x_2 \doteq -0.3826834, \quad x_3 \doteq -0.9238795$$

(5) Use the fact that $|f^{(4)}(t)| \leq 24$ for $-1 \leq t \leq 1$ to bound the error $|f(t) - P_3(t)|$ from problem (4).

$$|f(x) - P_3(x)| \leq \frac{L}{2^3} \quad \text{where} \quad L = \max_{[-1,1]} \left| \frac{f^{(4)}(x)}{4!} \right|$$

Here, $L = 24/4! = 1$ giving a bound

$$|f(x) - P_3(x)| \leq \frac{1}{2^3} = \frac{1}{8}$$

(6) Find the piece-wise linear interpolating function for the data set $\{(1,3), (1.5,2), (2,3.5)\}$.

$$\ell(x) = \begin{cases} -2x + 2, & 1 \leq x \leq 1.5 \\ 3x - 2.5, & 1.5 \leq x \leq 2 \end{cases}$$

(7) Find the natural cubic spline that interpolates the data in problem (6).

$$s(x) = \begin{cases} 5x^3 - 15x^2 + \frac{47}{4}x + \frac{5}{4}, & 1 \leq x \leq 1.5 \\ -5x^3 + 30x^2 - \frac{223}{4}x + 35, & 1.5 \leq x \leq 2 \end{cases}$$

(8) Use the trapazoid rule with two subintervals $T_2(f)$ and the trapazoid rule with 4 subintervals $T_4(f)$ to approximate

$$\int_0^2 \frac{1}{4 + x^2}\, dx.$$

Find the error for each case (compute the exact value using the Fundamental Thm of Calculus).

$$T_2(f) = \frac{31}{80} = 0.3875, \quad T_4(f) \doteq 0.3914$$

$$E_2^T(f) = \frac{\pi}{8} - T_2(f) \doteq 0.00520, \quad E_4^T(f) = \frac{\pi}{8} - T_4(f) \doteq 0.00130$$

(9) Repeat problem number (8) except using the Simpson's rules $S_2(f)$ and $S_4(f)$.

$$S_2(f) = \frac{47}{120} = 0.3917, \quad S_4(f) \doteq 0.3927$$

$$E_2^S(f) = \frac{\pi}{8} - S_2(f) \doteq 0.00100, \quad E_4^S(f) = \frac{\pi}{8} - S_4(f) \doteq -0.00000092$$

(10) Use the results from problems (8) and (9) to approximate

$$\int_0^2 \frac{1}{4+x^2}\, dx.$$

using the Richardson extrapolation methods $R_4(f)$ for both the trapazoid and the Simpson's rules. For the Trapazoid rule

$$R_4(f) = \frac{1}{3}(4T_4(f) -_2 (f)) \doteq 0.3927$$

For Simpson's rule

$$R_4(f) = \frac{1}{15}(16S_4(f) - S_2(f)) \doteq 0.3928$$

(11) Use Gaussian numerical integration $I_2(f)$ to approximate

$$\int_{-1}^1 \sqrt[3]{x}\, e^{-x}\, dx$$

Letting $f(x) = \sqrt[3]{x}\, e^{-x}$,

$$I_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \doteq -1.0158$$