Exam 2 Math 2306 sec. 51

Fall 2021

Name: (4 pts) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
Total (+4)	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas. **No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.**

Show all of your work on the paper provided to receive full credit.

1. Show that if a and b are different real numbers then the functions $f_1(x) = e^{ax}$ and $f_2(x) = e^{bx}$ are linearly independent on the interval $(-\infty, \infty)$.

We can use the Wronskian.

$$W(f_1,f_2)(x) = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix}$$

$$= e^{ax}(be^{bx}) - ae^{ax}(e^{bx})$$

$$= (b-a)\, e^{ax} e^{bx}$$

Since $e^{ax} \cdot e^{bx} > 0$ and $b \neq a$, (i.e. $b - a \neq 0$)

$W(f_1,f_2)(x) \neq 0$. They are therefore linearly independent.

2. An aquarium contains 50 gallons of water into which 10 pounds of salt is dissolved. Salt water containing $\frac{3}{10}$ lbs of salt per gallon is pumped in at a rate of 2 gallons per minute, and the well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in the tank at time t.

$$\frac{dA}{dt} = r_i C_i - r_o C_o$$

Here $r_i = 2$ gal/min $= r_o$

$C_i = \frac{3}{10} \frac{\text{lb}}{\text{gal}}$

The Volume $V(t) = 50$ gal since $r_i = r_o$.

So $C_o = \frac{A}{50}$

$$\frac{dA}{dt} = 2\left(\frac{3}{10}\right) - 2\left(\frac{A}{50}\right)$$

$$\frac{dA}{dt} + \frac{1}{25}A = \frac{3}{5} \qquad A(0) = 10 \text{ lb}.$$

$$P(t) = \frac{1}{25}, \quad \mu = e^{\int \frac{1}{25} dt} = e^{\frac{t}{25}}$$

$$\left(e^{\frac{t}{25}} A\right)' = \frac{3}{5} e^{t/25}$$

$$e^{t/25} A = \int \frac{3}{5} e^{\frac{t}{25}} dt = \frac{3}{5}(25) e^{\frac{t}{25}} + k$$

$$A = 15 + k e^{-\frac{t}{25}}$$

$A(0) = 10 \Rightarrow 15 + k = 10 \Rightarrow k = -5$

$$\boxed{\text{The amount of salt is } A = 15 - 5e^{-\frac{t}{25}} \text{ lbs}}$$

3. Find the general solution of the nonhomogeneous ODE.

$y'' - 3y' + 2y = 2e^x$

Find y_c: $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0 \Rightarrow m = 1$ or $m = 2$.

$y_1 = e^x$, $y_2 = e^{2x}$, $y_c = c_1 e^x + c_2 e^{2x}$

Find y_p: Using undetermined coef.

$y_p = (A e^x)x = Axe^x$

$y_p' = Ae^x + Axe^x$

$y_p'' = Ae^x + Ae^x + Axe^x$

$y_p'' - 3y_p' + 2y_p = 2e^x$

$2Ae^x + Axe^x - 3(Ae^x + Axe^x) + 2Axe^x = 2e^x$

$xe^x(A - 3A + 2A) + e^x(2A - 3A) = 2e^x$

$-Ae^x = 2e^x$

$A = -2$

So $y_p = -2xe^x$

The general solution

$$\boxed{y = c_1 e^x + c_2 e^{2x} - 2xe^x}$$

4. Find a particular solution of the nonhomogeneous ODE. A fundamental solution set $\{y_1, y_2\}$ for the associated homogeneous equation is given.

$x^2y'' - xy' = 4x^2, \quad x > 0 \qquad y_1 = 1, \quad y_2 = x^2$

In standard form, the ODE is

$$y'' - \frac{1}{x} y' = 4.$$

Using variation of parameters.

$$y_p = u_1 y_1 + u_2 y_2$$

$$g(x) = 4, \quad W = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x$$

$$u_1 = \int \frac{-g y_2}{W} \, dx = \int \frac{-4(x^2)}{2x} \, dx = -2\int x \, dx = -x^2$$

$$u_2 = \int \frac{g y_1}{W} \, dx = \int \frac{4 \cdot 1}{2x} \, dx = 2\int \frac{1}{x} \, dx = 2\ln x$$

$$y_p = -x^2(1) + (2\ln x)\, x^2$$

$$y_p = 2x^2 \ln x - x^2$$

* Since $-x^2$ can be combined with y_c

we could say

$$y_p = 2x^2 \ln x$$

5. Find the general solution of the homogeneous ODE.

$y'' - 4y' + 8y = 0$

The characteristic equation

$m^2 - 4m + 8 = 0$

$m^2 - 4m + 4 = -4$

$(m-2)^2 = -4 \Rightarrow m - 2 = \pm 2i \Rightarrow m = 2 \pm 2i$

$y_1 = e^{2x} \cos(2x), \quad y_2 = e^{2x} \sin(2x)$

$$\boxed{y = C_1 e^{2x} \cos(2x) + C_2 e^{2x} \sin(2x)}$$

6. Find the **form** of the general solution of the nonhomogeneous ODE when using the method of undetermined coefficients. Don't bother finding the coefficients, A, B, etc.

$y'' - 4y' + 8y = 5e^{2x} \cos(2x) + x$ Note y_c is above in # 5.

Let $g_1(x) = 5e^{2x} \cos(2x)$

$y_{p_1} = \left(A e^{2x} \cos(2x) + B e^{2x} \sin(2x)\right) x$

$$\boxed{y_{p_1} = Axe^{2x} \cos(2x) + Bxe^{2x} \sin(2x)}$$

Let $g_2(x) = x$

$$\boxed{y_{p_2} = Cx + D}$$

$y_p = y_{p_1} + y_{p_2}$