Exam 2 Math 2306 sec. 52

Fall 2021

Name: (4 pts) ______________________________

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
Total (+4)	

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas. **No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.**

Show all of your work on the paper provided to receive full credit.

1. Show that if a is any real number, then the functions $f_1(x) = e^{ax}$ and $f_2(x) = xe^{ax}$ are linearly independent on the interval $(-\infty, \infty)$.

Use the Wronskian

$$W(f_1, f_2)(x) = \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & e^{ax} + axe^{ax} \end{vmatrix}$$

$$= e^{2ax} + axe^{2ax} - axe^{2ax}$$

$$= e^{2ax}$$

Since $W = e^{2ax} > 0$ (i.e. $W \neq 0$), the functions are linearly independent.

2. A dying battery imparts a force of $E(t) = 10e^{-5t}$ volts to an RC circuit with a resistance of 2 ohms and a capacitance of 0.01 farads (i.e. $C = \frac{1}{100}$). If the initial charge on the capacitor $q(0) = 0$, determine the charge $q(t)$ on the capacitor.

$$Rq' + \frac{1}{C}q = E$$

We have $R = 2$, $C = \frac{1}{100}$, $E = 10e^{-5t}$

$$2q' + 100q = 10e^{-5t} \qquad q(0) = 0.$$

In standard form: $q' + 50q = 5e^{-5t}$

$$P(t) = 50, \qquad \mu = e^{\int 50\,dt} = e^{50t}$$

$$\frac{d}{dt}\left(e^{50t} q\right) = 5e^{-5t}e^{50t} = 5e^{45t}$$

$$e^{50t} q = \int 5e^{45t}\,dt = \frac{5}{45}e^{45t} + k$$

$$q = \frac{1}{9}e^{-5t} + ke^{-50t}$$

Use $q(0) = 0$ $\qquad 0 = \frac{1}{9}e^{0} + ke^{0}$

$$\Rightarrow k = -\frac{1}{9}$$

The charge $q(t) = \frac{1}{9}e^{-5t} - \frac{1}{9}e^{-50t}$.

3. Find the general solution of the nonhomogeneous ODE.

$$y'' + y' - 2y = 2e^x$$

Find y_c: $\quad m^2 + m - 2 = 0$

$(m-1)(m+2) = 0 \Rightarrow m = 1$ or $m = -2$

$y_1 = e^x, \quad y_2 = e^{-2x}, \quad y_c = c_1 e^x + c_2 e^{-2x}$

Find y_p: Using undetermined coef.

$g(x) = 2e^x, \quad y_p = (Ae^x)x = Axe^x$

$y_p' = Ae^x + Axe^x$

$y_p'' = Ae^x + Ae^x + Axe^x$

$y_p'' + y_p' - 2y_p = 2e^x$

$2Ae^x + Axe^x + (Ae^x + Axe^x) - 2Axe^x = 2e^x$

$xe^x(A + A - 2A) + e^x(2A + A) = 2e^x$

$3Ae^x = 2e^x$

$A = \frac{2}{3}$

$y_p = \frac{2}{3}xe^x$

The general solution

$$\boxed{y = c_1 e^x + c_2 e^{-2x} + \frac{2}{3}xe^x}$$

4. Find a particular solution of the nonhomogeneous ODE. A fundamental solution set $\{y_1, y_2\}$ for the associated homogeneous equation is given.

$x^2y'' - xy' = 4, \quad x > 0 \qquad y_1 = 1, \quad y_2 = x^2$

Variation of parameter

Standard form : $y'' - \frac{1}{x}y' = \frac{4}{x^2}$

$g(x) = \frac{4}{x^2}, \quad y_1 = 1, \quad y_2 = x^2$

$W = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x$

$y_p = u_1 y_1 + u_2 y_2$

$u_1 = \int \frac{-g y_2}{W} dx = \int \frac{-\frac{4}{x^2} \cdot x^2}{2x} dx = -2\int \frac{1}{x} dx$

$= -2\ln x$

$u_2 = \int \frac{g y_1}{W} dx = \int \frac{\frac{4}{x^2} \cdot 1}{2x} dx = 2\int x^{-3} dx$

$= -x^{-2}$

$y_p = (-2\ln x) \cdot 1 - x^{-2}(x^2)$

$= -2\ln x - 1$

$\boxed{y_p = -2\ln x - 1}$

* We can combine -1 with y_c and write

$y_p = -2\ln x$

5. Find the general solution of the homogeneous ODE.

$y'' + 6y' + 10y = 0$

Characteristic equation $m^2 + 6m + 10 = 0$

$m^2 + 6m + 9 = -1$

$(m+3)^2 = -1 \Rightarrow m + 3 = \pm i$

$m = -3 \pm i$

$y_1 = e^{-3x} \cos x$, $y_2 = e^{-3x} \sin x$

$$\boxed{y = C_1 e^{-3x} \cos x + C_2 e^{-3x} \sin x}$$

6. Find the **form** of the general solution of the nonhomogeneous ODE when using the method of undetermined coefficients. Don't bother finding the coefficients, A, B, etc.

$y'' + 6y' + 10y = 2x + 5e^{-3x}\sin(x)$ y above is y_c

Let $g_1(x) = 2x$, $\boxed{y_{p_1} = Ax + B}$

Let $g_2(x) = 5e^{-3x} \sin x$

$y_{p_2} = \left(C e^{-3x} \sin x + D e^{-3x} \cos x\right) x$

$$\boxed{y_{p_2} = Cx e^{-3x} \sin x + Dx e^{-3x} \cos x}$$

$$\boxed{y_p = y_{p_1} + y_{p_2}}$$