# Exam 3 Math 2306 sec. 51

Fall 2021

**Name:** (4 pts) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

**Signature:** ______________________________

| Problem | Points |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| Total (+4) | |

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.
**No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.**

Show all of your work on the paper provided to receive full credit.

**1.** Evaluate each Laplace transform.

(a) $\mathscr{L}\left\{e^{2t} - \cos(4t)\right\} = \mathscr{L}\{e^{2t}\} - \mathscr{L}\{\cos(4t)\} = \dfrac{1}{s-2} - \dfrac{s}{s^2+16}$

(b) $\mathscr{L}\left\{(t^2-3)^2\right\} = \mathscr{L}\{t^4\} - 6\mathscr{L}\{t^2\} + 9\mathscr{L}\{1\} = \dfrac{4!}{s^5} - 6\dfrac{2!}{s^3} + \dfrac{9}{s}$

$(t^2-3)^2 = t^4 - 6t^2 + 9$

(c) $\mathscr{L}\{\sin t\cos t\} = \frac{1}{2}\mathscr{L}\{\sin(2t)\} = \frac{1}{2}\dfrac{2}{s^2+4} = \dfrac{1}{s^2+4}$

$\sin t \cos t = \frac{1}{2}\sin(2t)$

**2.** Find the steady state **charge**, $q_p$, on the capacitor in the LRC series circuit described by the given equation.

$$q'' + 4q' + 5q = 8\sin t$$

(**Note:** The transient charge is $q_c = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$.)

5 Let $q_p = A\sin t + B\cos t$

This has no like terms in common with $q_c$.

4 $q_p' = A\cos t - B\sin t$

1 $q_p'' = -A\sin t - B\cos t$

$$q_p'' + 4q_p' + 5q_p = 8\sin t$$

$$\sin t\,(-A - 4B + 5A) + \cos t\,(-B + 4A + 5B) = 8\sin t$$

$4A - 4B = 8$

$4A + 4B = 0 \Rightarrow B = -A$

$8A = 8 \Rightarrow A = 1$ so $B = -1$

The steady state charge

$$\boxed{q_p = \sin t - \cos t}$$

**3.** Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\frac{3s}{s^2+4}\right\}$ $= 3\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = 3\cos(2t)$

(b) $\mathscr{L}^{-1}\left\{\frac{1}{s^6}\right\}$ $= \frac{1}{5!}\,\mathscr{L}^{-1}\left\{\frac{5!}{s^6}\right\} = \frac{1}{5!}t^5$

(c) $\mathscr{L}^{-1}\left\{\frac{s}{(s-1)(s+2)}\right\}$ $= \frac{1}{3}\,\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{2}{3}\,\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$= \frac{1}{3}e^{t} + \frac{2}{3}e^{-2t}$

$\frac{s}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$

$s = A(s+2) + B(s-1)$

Set $s=1$ $\quad 1 = 3A \quad A = \frac{1}{3}$

$s=-2 \quad -2 = -3B \quad B = \frac{2}{3}$

**4.** Suppose that $f$ is defined on $[0,\infty)$ and $\mathscr{L}\{f(t)\} = \frac{1}{\sqrt{s^2+1}}$. Evaluate

(a) $\mathscr{L}\{e^{2t}f(t)\}$ $= \frac{1}{\sqrt{(s-2)^2+1}}$

(b) $\mathscr{L}\{f(t-\pi)\mathscr{U}(t-\pi)\}$ $= \frac{e^{-\pi s}}{\sqrt{s^2+1}}$

$F(s) = \frac{1}{\sqrt{s^2+1}}$, $\quad F(s-2)$, $\quad e^{-\pi s}F(s)$

**5.** A 2 kg mass is attached to a spring with spring constant 32 N/m.

(a) If there is no damper, and a driving force $f(t) = \sin(\gamma t)$ is applied, what value of $\gamma$ will result in pure resonance?

Pure resonance is when $\gamma = \omega$

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{32}{2}} = 4$

$\boxed{\text{So } \gamma = 4}$

(b) If there is no driver, but a dashpot is added to induce damping of $\beta$ N per m/sec of velocity, what value of $\beta$ will result in critical damping?

$2x'' + \beta x' + 32x = 0$

Critical damping happens if $\beta^2 - 4mk = 0$

$\beta = \sqrt{4mk} = \sqrt{4 \cdot 2 \cdot 32} = \sqrt{8 \cdot 8 \cdot 4} = 16$ $\boxed{\beta = 16}$

**6.** Evaluate each Inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s^6}\right\} = \frac{1}{5!}(t-3)^5 u(t-3)$

from 3b

$f(t) = \frac{1}{5!}t^5$

(b) $\mathscr{L}^{-1}\left\{\frac{s}{s^2 - 2s + 5}\right\} = \mathscr{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}$

$= e^t \cos(2t) + \frac{1}{2} e^t \sin(2t)$

$s^2 - 2s + 1 + 4 = (s-1)^2 + 4$

$\frac{s}{(s-1)^2+4} = \frac{s-1}{(s-1)^2+4} + \frac{1}{(s-1)^2+4} = \frac{s-1}{(s-1)^2+4} + \frac{1}{2}\frac{2}{(s-1)^2+4}$