# Exam 3 Math 2306 sec. 52

Fall 2021

**Name:** (4 pts) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

**Signature:** ____________________

| Problem | Points |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| Total (+4) | |

INSTRUCTIONS: There are 6 problems worth 16 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.
**No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.**

Show all of your work on the paper provided to receive full credit.

**1.** Evaluate each Laplace transform.

(a) $\mathscr{L}\left\{\sin(2t)+2e^{-3t}\right\} = \mathscr{L}\{\sin(2t)\} + 2\mathscr{L}\{e^{-3t}\} = \dfrac{2}{s^2+4} + \dfrac{2}{s+3}$

(b) $\mathscr{L}\left\{(4-t^2)^2\right\} = 16\mathscr{L}\{1\} - 8\mathscr{L}\{t^2\} + \mathscr{L}\{t^4\} = \dfrac{16}{s} - 8\dfrac{2!}{s^3} + \dfrac{4!}{s^5}$

$(4-t^2)^2 = 16 - 8t^2 + t^4$

(c) $\mathscr{L}\left\{\sin(2t)\cos(2t)\right\} = \frac{1}{2}\mathscr{L}\{\sin(4t)\} = \frac{1}{2}\dfrac{4}{s^2+16} = \dfrac{2}{s^2+16}$

$\sin(2t)\cos(2t) = \frac{1}{2}\sin(4t)$

**2.** Find the steady state **charge**, $q_p$, on the capacitor in the LRC series circuit described by the given equation.

$$q'' + 2q' + 5q = 10\cos t$$

(**Note:** The transient charge is $q_c = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$.)

This has no like terms in common w/ $q_c$.

Let

5 $\quad q_p = A\cos t + B\sin t$

2 $\quad q_p' = -A\sin t + B\cos t$

1 $\quad q_p'' = -A\cos t - B\sin t$

$$q_p'' + 2q_p' + 5q_p =$$

$$(-A + 2B + 5A)\cos t + (-B - 2A + 5B)\sin t = 10\cos t$$

$$4A + 2B = 10$$

$$-2A + 4B = 0 \Rightarrow A = 2B$$

$$8B + 2B = 10 \Rightarrow B = 1,\ A = 2$$

The steady state Charge

$$\boxed{q_p = 2\cos t + \sin t}$$

**3.** Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\dfrac{2s}{s^2+9}\right\} = 2\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+9}\right\} = 2\cos(3t)$

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^8}\right\} = \dfrac{1}{7!}\mathscr{L}^{-1}\left\{\dfrac{7!}{s^8}\right\} = \dfrac{1}{7!}t^7$

(c) $\mathscr{L}^{-1}\left\{\dfrac{s}{(s+1)(s+3)}\right\} = \dfrac{-1}{2}\mathscr{L}^{-1}\left\{\dfrac{1}{s+1}\right\} + \dfrac{3}{2}\mathscr{L}^{-1}\left\{\dfrac{1}{s+3}\right\}$

$= \dfrac{-1}{2}e^{-t} + \dfrac{3}{2}e^{-3t}$

$$\frac{s}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$s = A(s+3) + B(s+1)$

$s=-1 \quad -1 = 2A \quad A = \frac{-1}{2}$

$s=-3 \quad -3 = -2B \quad B = \frac{3}{2}$

**4.** Suppose that $f$ is defined on $[0,\infty)$ and $\mathscr{L}\{f(t)\} = \dfrac{2}{\sqrt{s^2-1}}$. Evaluate

(a) $\mathscr{L}\{e^{2t}f(t)\} = \dfrac{2}{\sqrt{(s-2)^2-1}}$

(b) $\mathscr{L}\{f(t-\pi)\mathscr{U}(t-\pi)\} = \dfrac{2e^{-\pi s}}{\sqrt{s^2-1}}$

$F(s) = \dfrac{2}{\sqrt{s^2-1}} \qquad F(s-2), \qquad e^{-\pi s}F(s)$

**5.** A 2 kg mass is attached to a spring with spring constant 50 N/m.

(a) If there is no damper, and a driving force $f(t) = \sin(\gamma t)$ is applied, what value of $\gamma$ will result in pure resonance?

Pure resonance $\Rightarrow \gamma = \omega$

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{2}} = 5$ $\boxed{\text{The resonance frequency } \gamma = 5.}$

(b) If there is no driver, but a dashpot is added to induce damping of $\beta$ N per m/sec of velocity, what value of $\beta$ will result in critical damping?

$2x'' + \beta x' + 50x = 0$ critical damping $\Rightarrow \beta^2 - 4mk = 0$

$\beta = \sqrt{4mk} = \sqrt{4 \cdot 2 \cdot 50} = \sqrt{400} = 20$ $\boxed{\beta = 20}$

**6.** Evaluate each Inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$

$= e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)$

$\frac{s}{s^2+2s+5} = \frac{s+1-1}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} - \frac{2}{2}\frac{1}{(s+1)^2+4}$

(b) $\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s^8}\right\} = \frac{1}{7!}(t-3)^7 u(t-3)$

$f(t) = \frac{1}{7!}t^7$

from 36