Exam 4 Math 2306 sec. 51

Fall 2021

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
EC	
Total	

INSTRUCTIONS: There are 5 problems worth 20 points each plus one extra credit problem worth 10 points. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.

Show all of your work on the paper provided to receive full credit.

1. Suppose y solves the given initial value problem. $\dfrac{dy}{dt} + 2y = 3e^t, \quad y(0) = 4$

$\mathcal{L}\{y' + 2y\} = \mathcal{L}\{3e^t\}$

$sY(s) - y(0) + 2Y(s) = \dfrac{3}{s-1}$

$(s+2)Y(s) - 4 = \dfrac{3}{s-1}$

$(s+2)Y(s) = \dfrac{3}{s-1} + 4$

$Y(s) = \dfrac{3}{(s-1)(s+2)} + \dfrac{4}{s+2}$

Then the Laplace transform of y is (circle the correct $Y(s)$)

(a) $Y(s) = \dfrac{3}{(s-1)(s+2)} + \dfrac{4}{s+2}$

(b) $Y(s) = \dfrac{4}{(s-1)(s+3)} + \dfrac{2}{s+3}$

(c) $Y(s) = \dfrac{4}{(s+1)(s-3)} + \dfrac{2}{s-3}$

(d) $Y(s) = \dfrac{3}{(s+1)(s-2)} + \dfrac{4}{s-2}$

2. Solve the IVP using the Laplace transform.

$$y'' + 2y' + 17y = 0, \quad y(0) = 1, \quad y'(0) = 3$$

$Y = \mathcal{L}\{y\}$.

$\mathcal{L}\{y'' + 2y' + 17y\} = \mathcal{L}\{0\} = 0$

$s^2 Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{3} + 2(sY(s) - \underbrace{y(0)}_{1}) + 17 Y(s) = 0$

$(s^2 + 2s + 17) Y(s) - s - 3 - 2 = 0$

$(s^2 + 2s + 17) Y(s) = s + 5$

$Y(s) = \dfrac{s+5}{s^2 + 2s + 17}$

$b^2 - 4ac = 4 - 4 \cdot 1 \cdot 17 < 0$

Complete the square

$s^2 + 2s + 17 = s^2 + 2s + 1 - 1 + 17 = (s+1)^2 + 16$

$Y(s) = \dfrac{s+5}{(s+1)^2 + 4^2} = \dfrac{s+1}{(s+1)^2 + 4^2} + \dfrac{4}{(s+1)^2 + 4^2}$

Note: $\mathcal{L}^{-1}\left\{\dfrac{s+1}{(s+1)^2 + 4^2}\right\} = e^{-1t} \mathcal{L}^{-1}\left\{\dfrac{s}{s^2 + 4^2}\right\}$

and $\mathcal{L}^{-1}\left\{\dfrac{4}{(s+1)^2 + 4^2}\right\} = e^{-1t} \mathcal{L}^{-1}\left\{\dfrac{4}{s^2 + 4^2}\right\}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$\boxed{y(t) = e^{-t} \cos(4t) + e^{-t} \sin(4t)}$$

3. An RC series circuit with resistance 10 ohms and capacitance 1 millifarad ($\frac{1}{1000}$ farads) has initial charge on the capacitor $q(0) = 1$ Coulomb. After 1 second, a unit impulse $E(t) = \delta(t-1)$ volts is applied. Use the Laplace transform to find the charge $q(t)$ on the capacitor for $t > 0$. That is, solve the IVP

$$10\frac{dq}{dt} + 1000q = \delta(t-1), \quad q(0) = 1.$$

Recall that for any $a \geq 0$, $\mathscr{L}\{\delta(t-a)\} = e^{-as}$.

Divide by 10 $\quad q' + 100q = \frac{1}{10}\delta(t-1)$

Let $Q(s) = \mathcal{L}\{q(t)\}$.

$$\mathcal{L}\{q' + 100q\} = \mathcal{L}\{\tfrac{1}{10}\delta(t-1)\}$$

$$sQ(s) - \underset{1}{q(0)} + 100Q(s) = \frac{1}{10}e^{-1s}$$

$$(s+100)Q(s) = \frac{1}{10}e^{-s} + 1$$

$$Q(s) = \frac{1}{10}\frac{e^{-s}}{s+100} + \frac{1}{s+100}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+100}\right\} = e^{-100t}$$ and

$$\mathcal{L}^{-1}\{e^{-s}F(s)\} = f(t-1)u(t-1)$$

$$q(t) = \mathcal{L}^{-1}\{Q(s)\}$$

$$\boxed{q(t) = \frac{1}{10}e^{-100(t-1)}u(t-1) + e^{-100t}}$$

4. The following integrals formulas may be helpful. For any nonzero constant k,

$$\int x\cos(kx)\,dx = \frac{x}{k}\sin(kx)+\frac{1}{k^2}\cos(kx)+C, \qquad \int x\sin(kx)\,dx = -\frac{x}{k}\cos(kx)+\frac{1}{k^2}\sin(kx)+C.$$

Determine the Fourier series for $f(x) = \begin{cases} x, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$.

w/o symmetry.

$p = \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right|_{-\pi}^{0} = \frac{1}{\pi}\left(0-\frac{\pi^2}{2}\right) = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\cancel{\sin(nx)}\right|_{-\pi}^{0} + \frac{1}{n^2}\cos(nx)\Big|_{-\pi}^{0}$$

$$= \frac{1}{n^2\pi}\left[\cos 0 - \cos(-n\pi)\right] = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\cancel{\sin(nx)}^{\,0}\right|_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[0 - -\frac{(-\pi)}{n}\cos(-n\pi)\right] = \frac{-1}{n}(-1)^n = \frac{(-1)^{n+1}}{n}$$

$$\boxed{f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2\pi}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)}$$

5. Find the half-range sine series of $f(x) = 2$, for $0 < x < 1$.

$p = 1, \quad \frac{n\pi x}{p} = n\pi x$

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 2\sin(n\pi x)\,dx$$

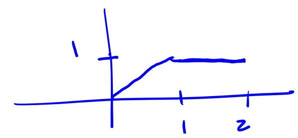
$$= 4\left[-\frac{1}{n\pi}\cos(n\pi x)\Big|_0^1\right.$$

$$= \frac{-4}{n\pi}\left[\cos(n\pi) - \cos(0)\right] = \frac{-4}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{4}{n\pi}\left(1 - (-1)^n\right)$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\left(1 - (-1)^n\right)\sin(n\pi x)}$$

(Extra Credit) Consider the function $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \le x < 2 \end{cases}$.

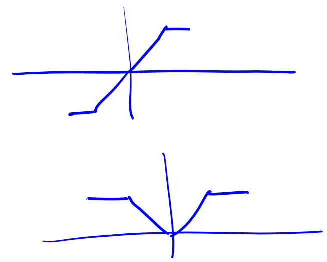


Below are three plots labeled (A), (B) and (C). For each plot, determine if it is the graph of the half-range cosine series of f, the half-range sine series f, or neither of these Fourier series of f.

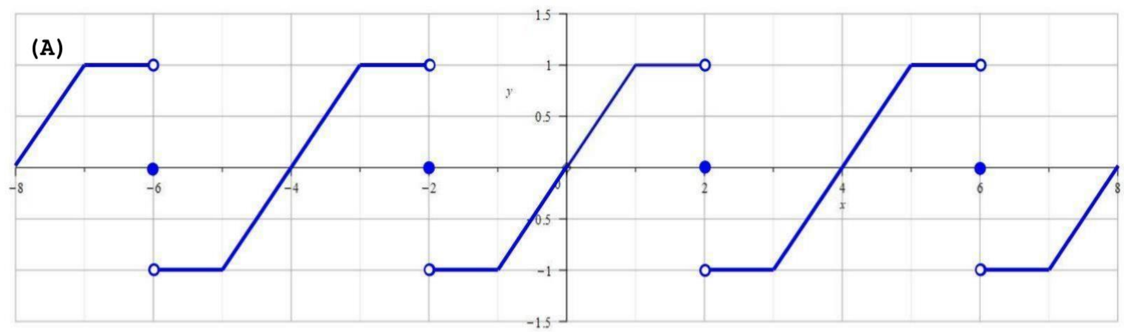
(A) Sine

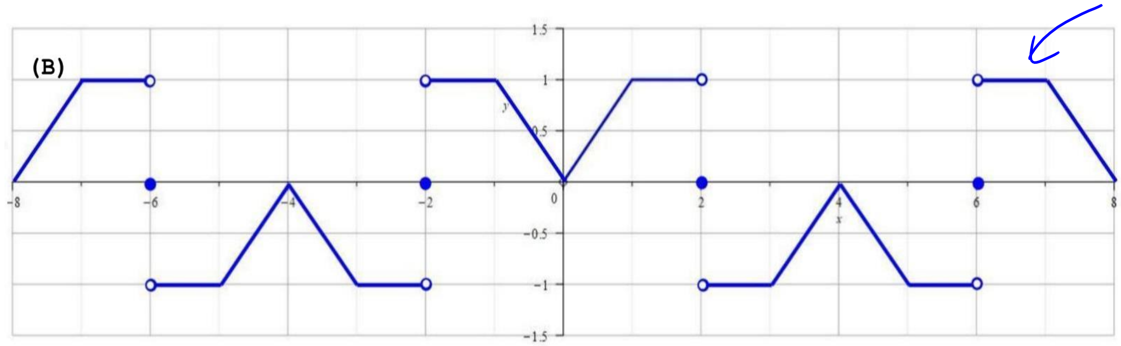
(B) neither

(C) Cosine



(You can write "cosine," "sine," or "neither.")





has period 8 should be 4