Exam 4 Math 2306 sec. 52

Fall 2021

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________

Problem	Points
1	
2	
3	
4	
5	
EC	
Total	

INSTRUCTIONS: There are 5 problems worth 20 points each plus one extra credit problem worth 10 points. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.

Show all of your work on the paper provided to receive full credit.

1. Suppose y solves the given initial value problem. $\dfrac{dy}{dt} - 2y = 3e^{-t}, \quad y(0) = 4$

$\mathcal{L}\{y' - 2y\} = \mathcal{L}\{3e^{-t}\}$

$sY(s) - y(0) - 2Y = \dfrac{3}{s+1}$

$(s-2)Y - 4 = \dfrac{3}{s+1}$

$(s-2)Y = \dfrac{3}{s+1} + 4$

$Y(s) = \dfrac{3}{(s+1)(s-2)} + \dfrac{4}{s-2}$

Then the Laplace transform of y is (circle the correct $Y(s)$)

(a) $Y(s) = \dfrac{3}{(s-1)(s+2)} + \dfrac{4}{s+2}$

(b) $Y(s) = \dfrac{4}{(s-1)(s+3)} + \dfrac{2}{s+3}$

(c) $Y(s) = \dfrac{4}{(s+1)(s-3)} + \dfrac{2}{s-3}$

(d) $Y(s) = \dfrac{3}{(s+1)(s-2)} + \dfrac{4}{s-2}$

2. Solve the IVP using the Laplace transform.

$$y'' + 2y' + 10y = 0, \quad y(0) = 1, \quad y'(0) = 2$$

$Y = \mathcal{L}\{y\}$

$\mathcal{L}\{y'' + 2y' + 10y\} = \mathcal{L}\{0\} = 0$

$s^2 Y - s\underset{1}{y(0)} - \underset{2}{y'(0)} + 2(sY - \underset{1}{y(0)}) + 10Y = 0$

$(s^2 + 2s + 10)Y(s) - s - 2 - 2 = 0$

$(s^2 + 2s + 10)Y = s + 4 \Rightarrow Y(s) = \dfrac{s+4}{s^2 + 2s + 10}$

$b^2 - 4ac = 4 - 4 \cdot 1 \cdot 10 < 0 \Rightarrow$ Complete the square

$s^2 + 2s + 1 - 1 + 10 = (s+1)^2 + 9 = (s+1)^2 + 3^2$

$Y(s) = \dfrac{s+4}{(s+1)^2 + 3^2} = \dfrac{s+1}{(s+1)^2 + 3^2} + \dfrac{3}{(s+1)^2 + 3^2}$

<u>Note</u>: $\mathcal{L}^{-1}\left\{\dfrac{s+1}{(s+1)^2 + 3^2}\right\} = e^{-1t}\mathcal{L}^{-1}\left\{\dfrac{s}{s^2 + 3^2}\right\}$ and

$\mathcal{L}^{-1}\left\{\dfrac{3}{(s+1)^2 + 3^2}\right\} = e^{-1t}\mathcal{L}^{-1}\left\{\dfrac{3}{s^2 + 3^2}\right\}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$\boxed{y(t) = e^{-t}\cos 3t + e^{-t}\sin 3t}$$

3. An LR series circuit with inductance of 2 henries and resistance 20 ohms has initial current of $i(0) = 1$ amp. After 1 second, a unit impulse $E(t) = \delta(t-1)$ volts is applied. Use the Laplace transform to find the current $i(t)$ in the circuit for $t > 0$. That is, solve the IVP

$$2\frac{di}{dt} + 20i = \delta(t-1), \quad i(0) = 1.$$

Recall that for any $a \geq 0$, $\mathscr{L}\{\delta(t-a)\} = e^{-as}$.

Let $I(s) = \mathscr{L}\{i(t)\}$. Divide by 2

$$i' + 10i = \frac{1}{2}\delta(t-1)$$

$$\mathscr{L}\{i' + 10i\} = \mathscr{L}\{\tfrac{1}{2}\delta(t-1)\} = \frac{1}{2}e^{-1s}$$

$$sI(s) - \underbrace{i(0)}_{1} + 10I(s) = \frac{1}{2}e^{-s}$$

$$(s+10)I(s) - 1 = \frac{1}{2}e^{-s}$$

$$I(s) = \frac{1}{2}\frac{e^{-s}}{s+10} + \frac{1}{s+10}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s+10}\right\} = e^{-10t}$$

and $\mathscr{L}^{-1}\{e^{-s}F(s)\} = f(t-1)u(t-1)$

$$i(t) = \mathscr{L}^{-1}\{I(s)\}$$

$$\boxed{i(t) = \frac{1}{2}e^{-10(t-1)}u(t-1) + e^{-10t}}$$

4. The following integrals formulas may be helpful. For any nonzero constant k,

$$\int x\cos(kx)\,dx = \frac{x}{k}\sin(kx)+\frac{1}{k^2}\cos(kx)+C, \qquad \int x\sin(kx)\,dx = -\frac{x}{k}\cos(kx)+\frac{1}{k^2}\sin(kx)+C.$$

Let $f(x) = x$, for $0 < x < 2\pi$. Determine the half-range cosine series of f.

$$p = 2\pi, \quad \frac{n\pi x}{p} = \frac{n\pi x}{2\pi} = \frac{nx}{2}$$

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi^2}{2} - 0\right] = 2\pi$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos\left(\frac{nx}{2}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos\left(\frac{nx}{2}\right)dx \qquad k = \frac{n}{2}, \quad \frac{1}{k} = \frac{2}{n}$$

$$= \frac{1}{\pi}\left[\frac{2x}{n}\underbrace{\sin\left(\frac{nx}{2}\right)}_{0}\Bigg|_0^{2\pi} + \frac{4}{n^2}\cos\left(\frac{nx}{2}\right)\Bigg|_0^{2\pi}\right.$$

$$= \frac{4}{n^2\pi}\left[\cos(n\pi) - \cos(0)\right]$$

$$= \frac{4}{n^2\pi}\left((-1)^n - 1\right)$$

$$\boxed{f(x) = \pi + \sum_{n=1}^{\infty}\frac{4}{n^2\pi}\left((-1)^n-1\right)\cos\left(\frac{nx}{2}\right)}$$

5. Find the half-range sine series of $f(x) = 1$, for $0 < x < \pi$.

$$p = \pi$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

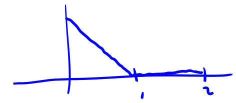
$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right|_0^{\pi}$$

$$= \frac{-2}{n\pi}\left[\cos(n\pi) - \cos(0)\right]$$

$$= \frac{-2}{n\pi}\left[(-1)^n - 1\right] = \frac{2}{n\pi}\left(1-(-1)^n\right)$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right)\sin(nx)}$$

(Extra Credit) Consider the function $f(x) = \begin{cases} 1-x, & 0 < x < 1 \\ 0, & 1 \le x < 2 \end{cases}$.

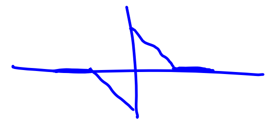


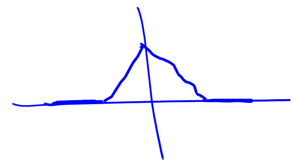
Below are three plots labeled (A), (B) and (C). For each plot, determine if it is the graph of the half-range cosine series of f, the half-range sine series f, or neither of these Fourier series of f.

(A) neither

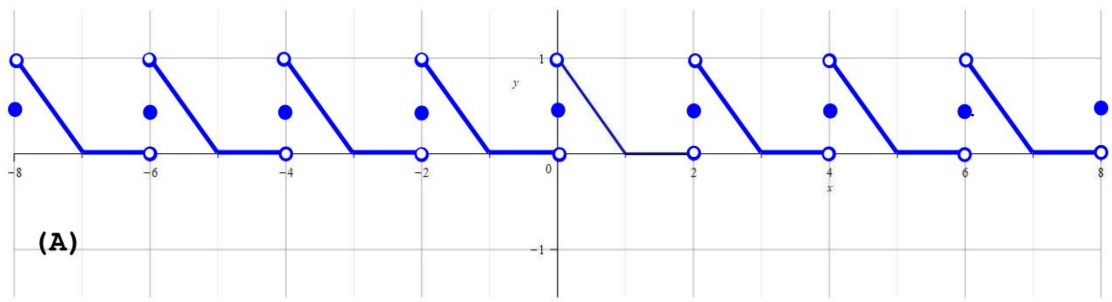
(B) Cosine

(C) Sine





(You can write "cosine," "sine," or "neither.")



← not even or odd!