February 10 Math 2306 sec. 51 Spring 2023

Section 5: First Order Equations Models and Applications

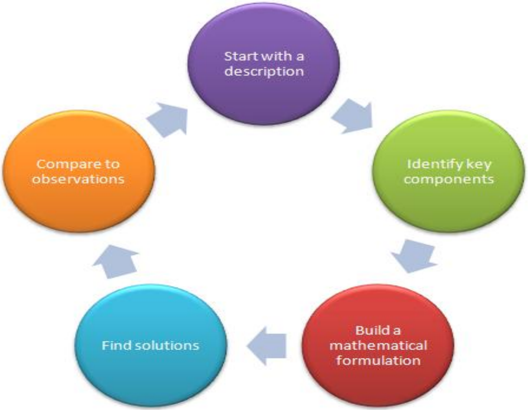


Figure: Mathematical Models give Rise to Differential Equations

Current Models

We have

Exponential Growth/Decay $\boxed{\dfrac{dP}{dt} = kP}$

RC-Series Circuit $\boxed{R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)}$

LR-Series Circuit $\boxed{L\dfrac{di}{dt} + Ri = E(t)}$

Classical Mixing $\boxed{\dfrac{dA}{dt} = r_i \cdot c_i - r_o \dfrac{A(t)}{V(0) + (r_i - r_o)t}}$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

$$\frac{dP}{dt} \propto P\,(M-P)$$

Difference between population and carrying capacity

$$\frac{dP}{dt} = kP(M-P)$$

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

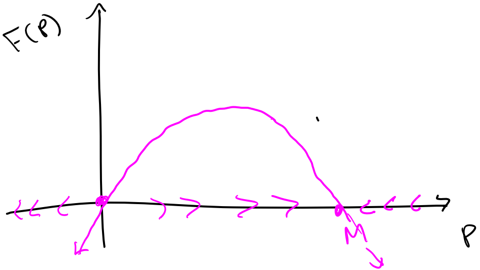
Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M - P), \quad k, M > 0$$

is called a **logistic growth equation.**
Solve this equation and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

$$\frac{dP}{dt} = F(P) = kMP - kP^2$$



$$\frac{dP}{dt} = kP(M-P), \quad = kMP - kP^2$$

Bernoulli

$$\frac{dP}{dt} - kMP = -kP^2$$

$$\frac{dy}{dt} + Q(t)y = f(t)y^n$$

$$u = P^{1-n} = P^{1-2} = P^{-1}$$

$n = 2$

$Q(t) = -kM$

$f(t) = -k$

u solves

$$\frac{du}{dt} + (1-n)Q(t)u = (1-n)f(t)$$

$$1-n = 1-2 = -1$$

$$\frac{du}{dt} + (-1)(-kM)u = (-1)(-k)$$

$$\frac{du}{dt} + kMu = k$$

$$Q_1(t) = kM, \quad \mu = e^{\int Q_1(t)\,dt} = e^{\int kM\,dt} = e^{kMt}$$

$$\frac{d}{dt}\left(e^{kMt} u\right) = k e^{kMt}$$

$$\int \frac{d}{dt}\left(e^{kMt} u\right) dt = \int k e^{kMt}\, dt$$

$$e^{kMt} u = k \frac{1}{kM} e^{kMt} + C$$

$$u = \frac{\frac{1}{M} e^{kMt} + C}{e^{kMt}}$$

$$u = \frac{1}{M} + C e^{-kMt}$$

$$u = P^{-1} = \frac{1}{P} \implies P = \frac{1}{u}$$

$$P(t) = \frac{1}{\frac{1}{M} + C e^{-kMt}} \cdot \frac{M}{M}$$

Clear fractions

$$P(t) = \frac{M}{1 + CM e^{-kMt}}$$

Suppose $P(0) = P_0 > 0$

Let $CM = C_1$

$$P(t) = \frac{M}{1 + C_1 e^{-kMt}}$$

$$P(0) = P_0 = \frac{M}{1 + C_1 e^{0}}$$

$$P_0 = \frac{M}{1 + C_1} \Rightarrow P_0 (1 + C_1) = M$$

$$P_0 + P_0 C_1 = M \Rightarrow P_0 C_1 = M - P_0$$

$$C_1 = \frac{M - P_0}{P_0}$$

$$P(t) = \frac{M}{1 + \frac{M - P_0}{P_0} e^{-kMt}} \cdot \frac{P_0}{P_0}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M - P_0)\, e^{-kMt}}}$$

This is the solution to the IVP

$$\frac{dP}{dt} = kP(M-P), \quad P(0) = P_0$$

In the limit

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{MP_0}{P_0 + (M-P_0)\, e^{-kMt}}$$

$$= \frac{MP_0}{P_0 + (M-P_0)\cdot 0} = \frac{MP_0}{P_0} = M$$

Logistic Modeling

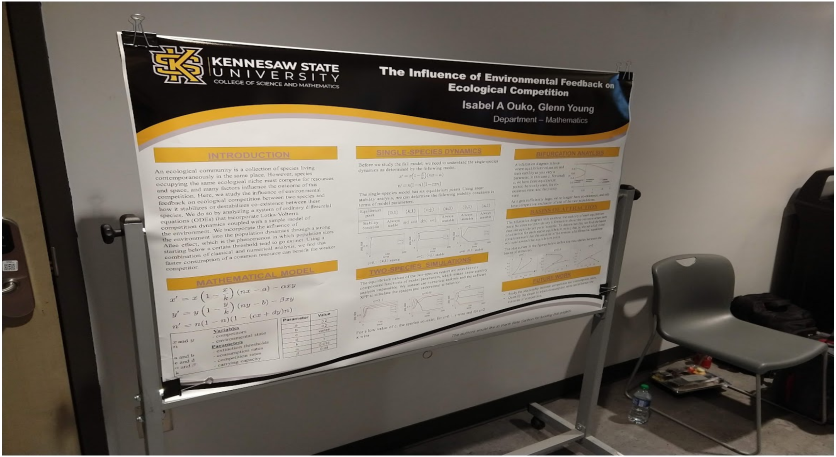
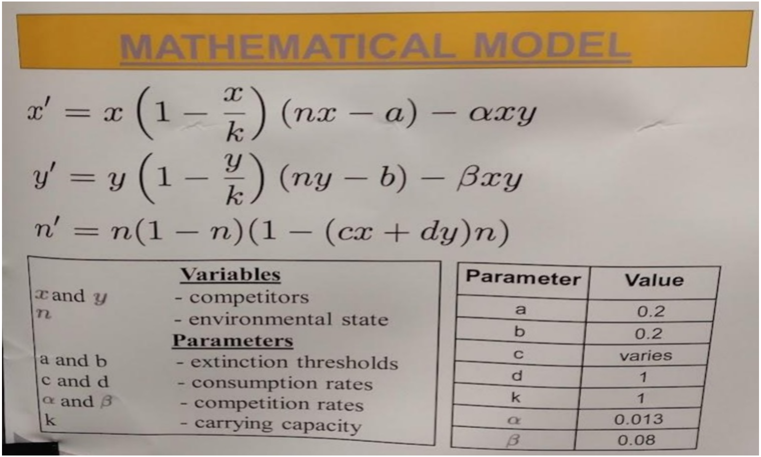


Figure: Poster of recent Birla Carbon scholar

Logistic Modeling



MATHEMATICAL MODEL

$$x' = x\left(1 - \frac{x}{k}\right)(nx - a) - \alpha xy$$

$$y' = y\left(1 - \frac{y}{k}\right)(ny - b) - \beta xy$$

$$n' = n(1 - n)(1 - (cx + dy)n)$$

Variables

x and y	- competitors
n	- environmental state

Parameters

a and b	- extinction thresholds
c and d	- consumption rates
α and β	- competition rates
k	- carrying capacity

Parameter	Value
a	0.2
b	0.2
c	varies
d	1
k	1
α	0.013
β	0.08

Figure: The species equations include an extended logistic term with threshold and competition.