February 10 Math 2306 sec. 52 Spring 2023

Section 5: First Order Equations Models and Applications

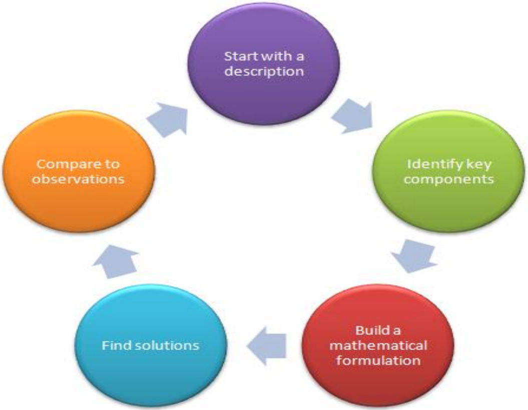


Figure: Mathematical Models give Rise to Differential Equations

Current Models

We have

Exponential Growth/Decay $$\boxed{\frac{dP}{dt} = kP}$$

RC-Series Circuit $$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

LR-Series Circuit $$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

Classical Mixing $$\boxed{\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A(t)}{V(0) + (r_i - r_o)t}}$$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

$$\frac{dP}{dt} \propto P(M-P)$$

difference between population and carrying capacity

$$\frac{dP}{dt} = k\,P(M-P) \quad , \quad k\text{- constant}$$

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

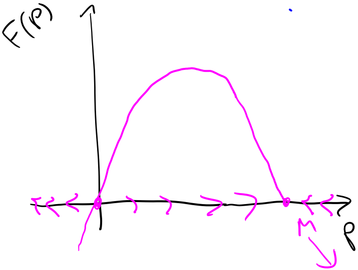
The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

$$\frac{dP}{dt} = F(P) = kMP - kP^2$$



$$\frac{dP}{dt} = kP(M-P), = kMP - kP^2$$

$$\frac{dP}{dt} - kMP = -kP^2$$

Let $u = P^{1-n} = P^{1-2} = P^{-1}$

$1-n = -1$

u solves

$$\frac{du}{dt} + (1-n)Q(t)\,u = (1-n)f(t)$$

$$\frac{du}{dt} + (-1)(-kM)u = (-1)(-k)$$

$$\frac{du}{dt} + kMu = k$$

Bernoulli

$$\frac{dy}{dt} + Q(t)y = f(t)y^n$$

$Q(t) = -kM$

$f(t) = -k$

$n = 2$

$$Q_1(t) = kM \qquad \mu = e^{\int Q_1(t)\,dt}$$

$$= e^{\int kM\,dt} = e^{kMt}$$

$$\frac{d}{dt}\left(e^{kMt} u\right) = k e^{kMt}$$

$$e^{kMt} u = \int k e^{kMt}\,dt$$

$$e^{kMt} u = k\left(\frac{1}{kM}\right) e^{kMt} + C$$

$$u = \frac{\frac{1}{M} e^{kMt} + C}{e^{kMt}}$$

$$u = \frac{1}{M} + C e^{-kMt}$$

$$= \frac{1 + CMe^{-kMt}}{M}, \quad \text{let } C_1 = CM$$

$$u = \frac{1 + C_1 e^{-kMt}}{M}$$

$$u = P^{-1} = \frac{1}{P} \Rightarrow P = \frac{1}{u}$$

$$P = \frac{M}{1 + C_1 e^{-kMt}}$$

Let $P(0) = P_0 > 0$

$$P(0) = \frac{M}{1 + C_1 e^0} = P_0$$

solve for C_1

$$M = P_0(1 + C_1)$$

$$= P_0 + P_0 C_1$$

$$\Rightarrow \quad P_0 C_1 = M - P_0$$

$$C_1 = \frac{M - P_0}{P_0}$$

$$P(t) = \frac{M}{1 + \frac{M - P_0}{P_0} e^{-kMt}} \cdot \frac{P_0}{P_0}$$

Clear fractions

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}}$$

This is the solution to the IVP

$$\frac{dP}{dt} = kP(M-P) \qquad P(0) = P_0$$

To see the long time solution

take $t \to \infty$.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{MP_0}{P_0 + (M-P_0)e^{-k \cdot Mt}}$$

$$= \frac{MP_0}{P_0 + (M - P_0)\cdot 0} = \frac{MP_0}{P_0} = M$$

Logistic Modeling

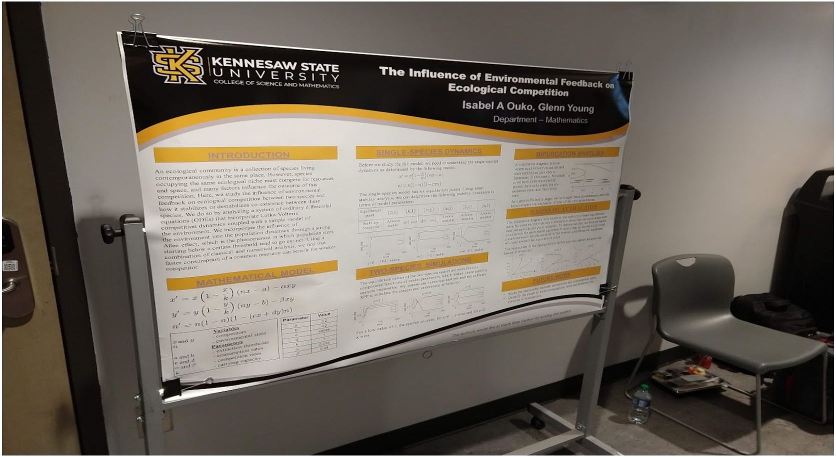
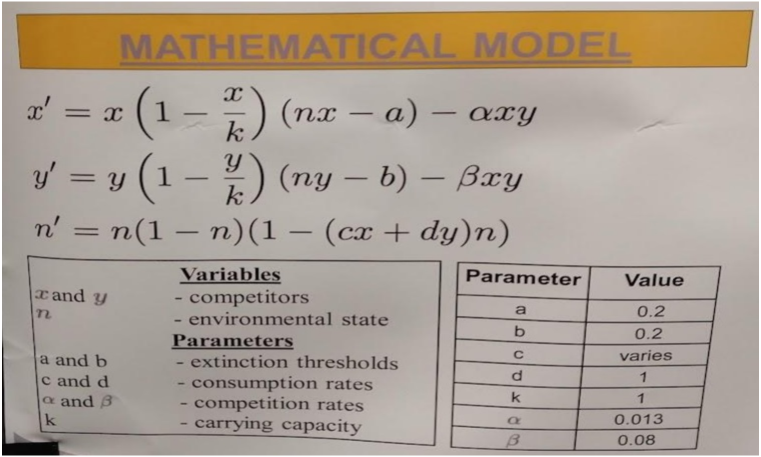


Figure: Poster of recent Birla Carbon scholar

Logistic Modeling



MATHEMATICAL MODEL

$$x' = x\left(1 - \frac{x}{k}\right)(nx - a) - \alpha xy$$

$$y' = y\left(1 - \frac{y}{k}\right)(ny - b) - \beta xy$$

$$n' = n(1 - n)(1 - (cx + dy)n)$$

	Variables
x and y	- competitors
n	- environmental state
	Parameters
a and b	- extinction thresholds
c and d	- consumption rates
α and β	- competition rates
k	- carrying capacity

Parameter	Value
a	0.2
b	0.2
c	varies
d	1
k	1
α	0.013
β	0.08

Figure: The species equations include an extended logistic term with threshold and competition.