February 11 Math 3260 sec. 52 Spring 2022

Section 1.8: Intro to Linear Transformations

Recall that the product $A\mathbf{x}$ is a linear combination of the columns of A. This turns out to be a vector.

If the columns of A are vectors in $\mathbb{R}^m$, and there are n of them, then

- A is an $m \times n$ matrix,
- the product $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbb{R}^n$, and
- the vector $\mathbf{b} = A\mathbf{x}$ is a vector in $\mathbb{R}^m$.

Remark: We can think of a matrix A as an **object that acts** on vectors $\mathbf{x}$ in $\mathbb{R}^n$ (via the product $A\mathbf{x}$) to produce vectors $\mathbf{b}$ in $\mathbb{R}^m$.

Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition: A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Remark: Such a transformation can be called a **function** or a **mapping**. It will take a vector as an input and spit out a vector as an output.

Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Function Notation: If a transformation T takes a vector $\mathbf{x}$ in $\mathbb{R}^n$ and maps it to a vector $T(\mathbf{x})$ in $\mathbb{R}^m$, we can write

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

which reads "T maps $\mathbb{R}^n$ into $\mathbb{R}^m$."

And we can write

$$\mathbf{x} \mapsto T(\mathbf{x})$$

which reads "$\mathbf{x}$ maps to T of $\mathbf{x}$."

The following vertically stacked notation is often used:

$$\begin{aligned} T : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \mathbf{x} &\mapsto T(\mathbf{x}) \end{aligned}$$

Key Terms

For $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

- $\mathbb{R}^n$ is the **domain**, and
- $\mathbb{R}^m$ is called the **codomain.**
- For $\mathbf{x}$ in the domain, $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ under T. (We can call $\mathbf{x}$ a **pre-image** of $T(\mathbf{x})$.)
- The collection of all images is called the **range.**
- If $T(\mathbf{x})$ is defined by multiplication by the $m \times n$ matrix A, we may denote this by $\mathbf{x} \mapsto A\mathbf{x}$.

Matrix Transformation Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$. Define the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by the mapping $T(\mathbf{x}) = A\mathbf{x}$.

(a) Find the image of the vector $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ under T.

We want to find the vector $T(\vec{u})$.

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ 6 \end{bmatrix}$$

$\begin{bmatrix} -8 \\ -10 \\ 6 \end{bmatrix}$ is the image of $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ under T.

Example Continued...

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}, \qquad \begin{array}{c} T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ \vec{x} \mapsto A\mathbf{x} \end{array}$$

(b) Determine a vector $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$.

we can state this as find a vector $\vec{x}$ such that $T(\vec{x}) = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$, i.e. $A\vec{x} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$.

The matrix eqn is

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$$

We can use an augmented matrix

$$\begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{array}{l} x_1 = 2 \\ x_2 = -2. \end{array}$$

Hence a preimage for $\begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$ under T

is $\vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

Example Continued...

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}, \qquad \begin{array}{rl} T : \mathbb{R}^n & \longrightarrow \mathbb{R}^m \\ \vec{x} & \mapsto A\mathbf{x} \end{array}$$

(c) Determine if $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in the range of T.

This is asking if $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an output $T(\vec{x})$ for some $\vec{x}$ in the domain. Is there an $\vec{x}$ in $\mathbb{R}^2$ such that $T(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The matrix equation is

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot column

$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is inconsistent.

Hence $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not in the range of T.

Linear Transformations

Definition: A transformation T is **linear** provided

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

Remark 1: I made a big deal in section 1.4 about these two properties. These properties are key when we use the term **Linear**.

Remark 2: Every matrix transformation (e.g. $\mathbf{x} \mapsto A\mathbf{x}$) is a linear transformation. And it turns out that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be expressed in terms of matrix multiplication.

A Theorem About Linear Transformations:

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}, \quad \text{and}$$

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for scalars c, and d and vectors $\mathbf{u}$ and $\mathbf{v}$.

And in fact

$$T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \cdots + c_kT(\mathbf{u}_k).$$

Remark: This says that the image of a linear combination is the linear combination of the images.

Example

Let r be a nonzero scalar. The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = r\mathbf{x}$$

is a linear transformation[1].

Example: Show that T is a linear transformation.

We need to show that T satisfies the two properties. Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^2$ and c in $\mathbb{R}$.

$$T(\vec{u}+\vec{v}) = r(\vec{u}+\vec{v}) = r\vec{u} + r\vec{v} = T(\vec{u}) + T(\vec{v})$$

[1] It's called a **contraction** if $0 < r < 1$ and a **dilation** when $r > 1$

Also $T(c\vec{u}) = r(c\vec{u}) = rc\vec{u} = cr\vec{u}$

$= c(r\vec{u}) = cT(\vec{u})$.

T satisfies both properties; Hence

T is a linear transformation.

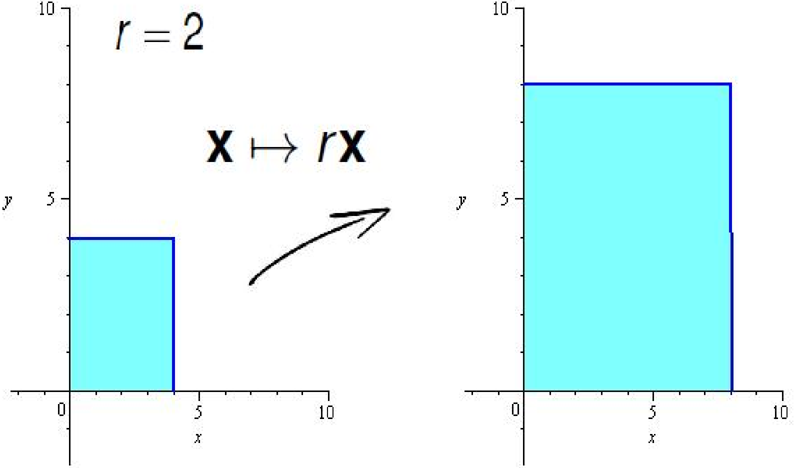

Figure: Geometry of dilation $\mathbf{x} \mapsto 2\mathbf{x}$. The 4 by 4 square maps to an 8 by 8 square.