February 12 Math 3260 sec. 51 Spring 2024

Section 1.9: The Matrix for a Linear Transformation

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]\,.$$

The matrix A is called the **standard matrix** for the linear transformation T.

Onto and One to One

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of **at most one** $\mathbf{x}$ in $\mathbb{R}^n$.

Some Theorems about *Onto* and *One to One*

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Example

Consider the linear transformation
$$\begin{aligned} T : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x_1, x_2, x_3) &\mapsto (x_3, x_1 + x_2) \end{aligned}$$

Determine the set of all preimages[1] of $\mathbf{0}$. State the solution as a span.

The preimages of $\mathbf{0}$ are all vectors $\mathbf{x}$ in $\mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{0}$. We found the standard matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The preimages of the zero vector are solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Using row reduction, we found this to be

$$\text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

[1] This actually has a special name. The set of all preimages of the zero vector is called the *kernel* of T.

Example

Consider the linear transformation
$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x_1, x_2, x_3) \mapsto (x_3, x_1 + x_2)$$

Is T one to one? Is T onto? The standard matrix

$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ with rref $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$A\vec{x} = \vec{0}$ has nontrivial solutions.
Hence T is not one to one.

A has 2 pivot positions, the columns
span $\mathbb{R}^2$. Hence T is onto.

Example

Consider the linear transformation $\begin{array}{c} T : \mathbb{R}^2 \to \mathbb{R}^3 \\ (x_1, x_2) \mapsto (x_2, 0, -x_1) \end{array}$. Determine whether T is one to one, onto, neither or both.

An option is to find the standard matrix A.

we need $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

$T(\vec{e}_1) = T(1,0) = (0,0,-1)$

$T(\vec{e}_2) = T(0,1) = (1,0,0)$.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$. We can consider the homogeneous equation $T(\vec{x}) = \vec{0}$.

$A\vec{x} = \vec{0}$. The augmented matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$$

The system has only the trivial solution.

T is one to one.

A only has two columns, so its columns can't span $\mathbb{R}^3$

T is not onto.