February 12 Math 3260 sec. 52 Spring 2024

Section 1.9: The Matrix for a Linear Transformation

Theorem

Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m\times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x}\in\mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n\times n$ identity matrix I_n. That is,

$$A = \left[T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)\right].$$

The matrix A is called the **standard matrix** for the linear transformation T.

Onto and One to One

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of at least one **x** in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

Some Theorems about *Onto* and *One to One*

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Example

Consider the linear transformation
$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x_1, x_2, x_3) \mapsto (x_3, x_1 + x_2)$$

Determine the set of all preimages[1] of $\mathbf{0}$. State the solution as a span.

We can restate this as find all $\vec{x}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$. We can use the standard matrix, A, and solve $A\vec{x} = \vec{0}$.

$$A = [T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)].$$

$$T(\vec{e}_1) = T(1,0,0) = (0, 1+0) = (0,1)$$

$$T(\vec{e}_2) = T(0,1,0) = (0, 0+1) = (0,1)$$

[1] This actually has a special name. The set of all preimages of the zero vector is called the *kernel* of T.

$T(\vec{e}_3) = T(0,0,1) = (1, 0+0) = (1,0)$

$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Considering $A\vec{x} = \vec{0}$

$$[A \ \vec{0}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$A\vec{x} = \vec{0}$ if $x_1 = -x_2$
x_2 - is free
$x_3 = 0$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

The set of preimages of $\vec{0}$ is
$$\text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Example

Consider the linear transformation
$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x_1, x_2, x_3) \mapsto (x_3, x_1 + x_2)$$

Is T one to one? Is T onto?

we found that $T(\vec{x}) = \vec{0}$ has nontrivial solutions.

Hence $\boxed{T \text{ is } \underline{\text{not}} \text{ one to one}}$.

we found standard matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

A has two pivot positions, so its columns span $\mathbb{R}^2$. $\boxed{\text{so } T \text{ is onto.}}$

Example

Consider the linear transformation $\begin{array}{c} T : \mathbb{R}^2 \to \mathbb{R}^3 \\ (x_1, x_2) \mapsto (x_2, 0, -x_1) \end{array}$. Determine whether T is one to one, onto, neither or both.

We can find the standard matrix, A.

$A = [T(\vec{e}_1)\ T(\vec{e}_2)]$.

$T(\vec{e}_1) = T(1,0) = (0,0,-1)$

$T(\vec{e}_2) = T(0,1) = (1,0,0)$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$. Consider $T(\vec{x}) = \vec{0}$,
i.e., $A\vec{x} = \vec{0}$.

$$[A\ \vec{0}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

There are no nontrivial solutions.

$\boxed{\text{Hence } T \text{ is one to one.}}$

A has only two columns, so its columns can't span $\mathbb{R}^3$. $\boxed{T \text{ is not onto.}}$