February 13 Math 2306 sec. 51 Spring 2023

Section 6: Linear Equations Theory and Terminology

Consider the second order, linear ODE

$$x^2y'' - xy' + y = 1.$$

It is easy to show that $y = x + 1$ is a solution.

$y' = 1, \quad y'' = 0$

$x^2y'' - xy' + y = x^2(0) - x(1) + (x+1) \overset{?}{=} 1$

$-x + x + 1 \overset{?}{=} 1$

$1 = 1$

$x^2y'' - xy' + y = 1$

Here are 10 more solutions to this ODE!

$$y = 1 \qquad y = x\ln x + 1$$

$$y = 3x - x\ln x + 1 \qquad y = 7x\ln x + 8x + 1$$

$$y = 1 - 4x\ln\sqrt{x} \qquad y = 5x\ln\left(\frac{1}{x}\right) + 1 - x$$

$$y = 16x + x\ln x + 1 \qquad y = 1 - x\ln x^3$$

$$y = 16x\ln x^2 + \frac{2}{7}x + 1 \qquad y = \frac{x}{3} + x\ln x^7 + 1$$

An IVP

Consider the IVP

$$x^2y'' - xy' + y = 1, \quad y(1) = 1, \quad y'(1) = -1$$

Not one of the eleven solutions that I showed solve this IVP!

This raises some questions.

- What do mean when we talk about **solving** an ODE or an IVP?
- How do we know when we're done **solving** an ODE?
- Is there something we would call **THE solution**?

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

The Principle of Superposition (homogeneous ode)

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

Assume a_i are continuous and $a_n(x)\neq 0$ for all x in I.

Theorem: If y_1, $y_2,\ldots,y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1,\ldots,c_k$.

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I. \tag{1}$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

NOTE: Taking all of the c's to be zero will **always** satisfy equation (1). The set of functions is linearly **independent** if taking all of the c's equal to zero is the **only** way to make the equation true.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Suppose $c_1 f_1(x) + c_2 f_2(x) = 0$ for all x in $\mathbb{R}$.

$c_1 \sin x + c_2 \cos x = 0$

We'll assume this is true. We need to show that $c_1 = 0$ and $c_2 = 0$. The equation is true for all x, so it's true when $x = 0$. When $x = 0$, the equation is

$$c_1 \sin 0 + c_2 \cos 0 = 0$$

$\sin 0 = 0$, $\cos 0 = 1$

$c_1(0) + c_2(1) = 0 \implies c_2 = 0$

The equation is also true when $x = \frac{\pi}{2}$.

When $x = \frac{\pi}{2}$, the equation is

$$c_1 \sin\frac{\pi}{2} + c_2 \cos\frac{\pi}{2} = 0$$

$$c_1 \sin\frac{\pi}{2} = 0 \qquad \sin\frac{\pi}{2} = 1$$

$$c_1(1) = 0$$

$$c_1 = 0$$

That is, $c_1 = 0$ and $c_2 = 0$. This confirms that f_1 and f_2 are linearly independent

Determine if the set is Linearly Dependent or Independent on $(-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

Let's consider the equation

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all } x$$

$$c_1 x^2 + c_2(4x) + c_3(x - x^2) = 0$$

The x^2 will cancel if $c_1 = c_3$

The x will cancel if $c_2 = \frac{-1}{4} c_3$

Try $c_3 = 1$. Then take $c_1 = 1$ and $c_2 = \frac{-1}{4}$

$$c_1 x^2 + c_2 (4x) + c_3 (x - x^2) = 0$$

$$1x^2 + \left(\frac{-1}{4}\right)(4x) + 1(x - x^2) \stackrel{?}{=} 0$$

$$x^2 - x + x - x^2 \stackrel{?}{=} 0$$

$$0 = 0$$

We found a set of numbers (c's) with at least one being nonzero that made the equation $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ true.

This set of functions is linearly dependent.

Linear Dependence Relation

An equation with at least one c nonzero, such as

$$f_1(x) - \frac{1}{4}f_2(x) + f_3(x) = 0$$

from this last example is called a **linear dependence relation** for the functions $\{f_1, f_2, f_3\}$.

Definition of Wronskian

Definition: Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

2 functions $\Longrightarrow$ 2x2 matrix

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x\,(-\sin x) - \cos x\,(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\sin^2 x + \cos^2 x)$$

$$= -1$$

$$W(\sin x, \cos x)(x) = -1$$