February 14 Math 3260 sec. 51 Spring 2022

Section 1.9: The Matrix for a Linear Transformation

Definition: A transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** provided for every vector $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and every scalar c

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v}), \quad \text{and}$$

$$T(c\mathbf{u}) = cT(\mathbf{u}).$$

Remark: We know that a mapping defined by matrix multiplication $\mathbf{x} \mapsto A\mathbf{x}$ is a linear transformation. In fact, every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be realized in terms of matrix multiplication.

Elementary Vectors

Elementary Vectors: We'll use the notation $\mathbf{e}_i$ to denote the vector in $\mathbb{R}^n$ having a 1 in the i^{th} position and zero everywhere else.

e.g. in $\mathbb{R}^2$ the elementary vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

in $\mathbb{R}^3$ they would be

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and so forth.

Note that in $\mathbb{R}^n$, the elementary vectors are the columns of the identity I_n.

Example: Matrix of Linear Transformation

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ be a linear transformation, and suppose

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}.$$

Use the fact that T is linear, and the fact that for each $\mathbf{x}$ in $\mathbb{R}^2$ we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\,\mathbf{e}_1 + x_2\mathbf{e}_2$$

to find a matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^2.$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

Let's find $T(\vec{x})$ for arbitrary $\vec{x}$ in $\mathbb{R}^2$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2$$

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2)$$

$$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix}$ satisfies $T(\vec{x}) = A\vec{x}$.

As $\vec{x}$ was arbitrary, this is true for all $\vec{x}$ in $\mathbb{R}^2$

Note $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = \left[T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n) \right].$$

The matrix A is called the **standard matrix** for the linear transformation T.

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the scaling trasformation (contraction or dilation for $r > 0$) defined by

$$T(\mathbf{x}) = r\mathbf{x}, \quad \text{for positive scalar } r.$$

Find the standard matrix for T.

The standard matrix $A = [T(\vec{e}_1) \quad T(\vec{e}_2)]$

Note, the domain is $\mathbb{R}^2$.

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T(\vec{e}_1) = r\vec{e}_1 = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$

$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $T(\vec{e}_2) = r\vec{e}_2 = r\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$

So $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$.

Check: $T(\vec{x}) = r\vec{x} = r\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 + 0x_2 \\ 0x_1 + rx_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}$$

as expected!

Example: Shear Transformation

Find the standard matrix for the linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathbf{e}_2$ to $\mathbf{e}_2 - \frac{1}{2}\mathbf{e}_1$ and leaves $\mathbf{e}_1$ unchanged.

Calling this transformation T,

$$T(\vec{e}_1) = \vec{e}_1 \quad \text{and} \quad T(\vec{e}_2) = \vec{e}_2 - \frac{1}{2}\vec{e}_1$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

So

$$A = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Example: Shear Transformation

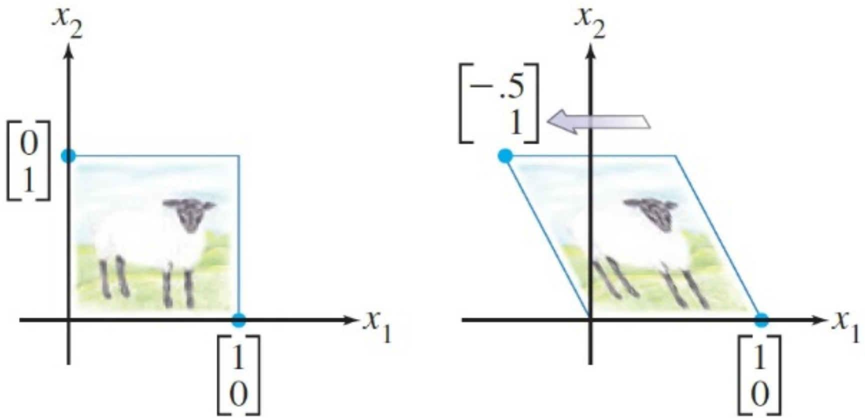
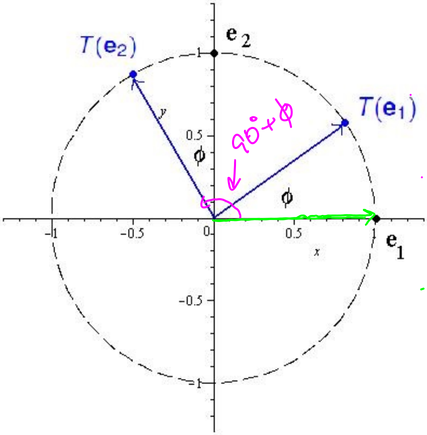


Figure: The unit square under the transformation $\mathbf{x} \mapsto \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \mathbf{x}$.

Example: Rotation

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation transformation that rotates each point in $\mathbb{R}^2$ counter clockwise about the origin through an angle ϕ. Find the standard matrix for T.



Using some basic trigonometry, the points on the unit circle

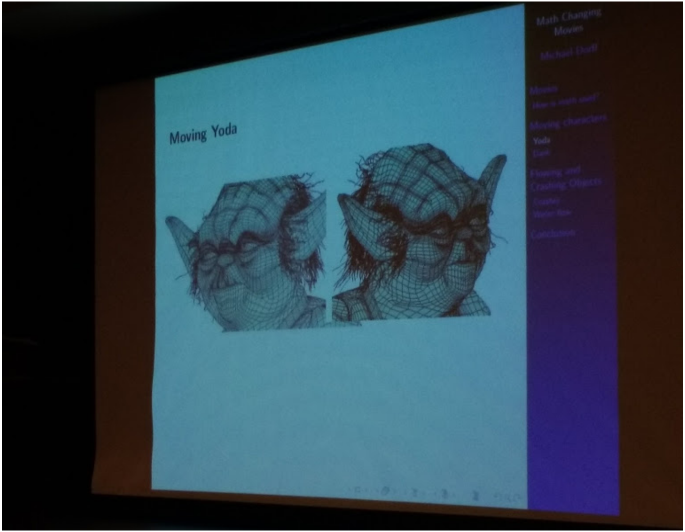
$$T(\mathbf{e}_1) = (\cos\phi, \sin\phi)$$

$$T(\mathbf{e}_2) = (\cos(90^\circ + \phi), \sin(90^\circ + \phi))$$

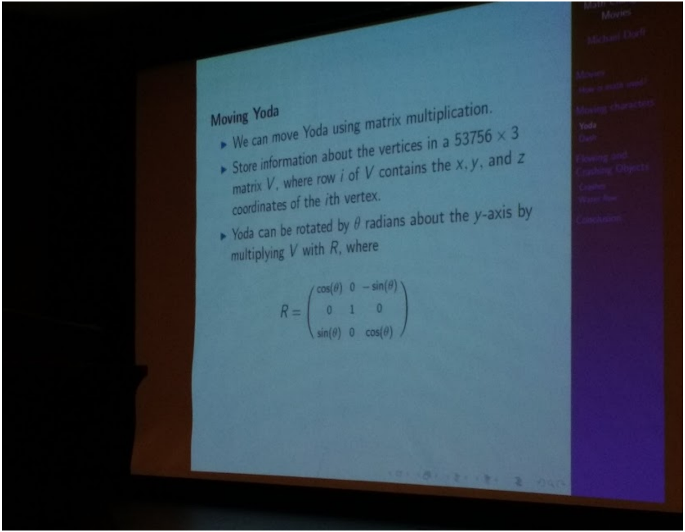
$$= (-\sin\phi, \cos\phi)$$

So $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$.

Rotation in Animation



Rotation in Animation



Example[1]

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection transformation that projects each point onto the x_1 axis

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

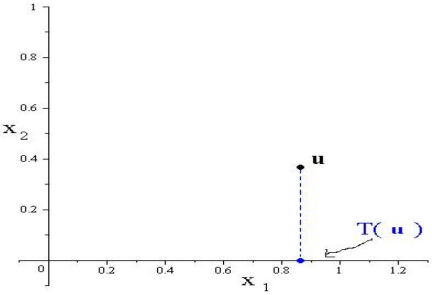
Find the standard matrix for T.

$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$T(\vec{x}) = A\vec{x}$ where

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



[1] See pages 77–80 in Lay for matrices associated with other geometric tranformation on $\mathbb{R}^2$

The Property **Onto**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an **onto** transformation, then the equation

$$T(\mathbf{x}) = \mathbf{b}$$

is always solvable. If T is a linear transformation with standard matrix A, then this is equivalent to saying $A\mathbf{x} = \mathbf{b}$ is always consistent.

Determine if the transformation is onto.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

We need to determine whether each vector $\vec{b}$ in $\mathbb{R}^m$ is an output for T.

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, it's 2×3. So, $T: \mathbb{R}^3 \to \mathbb{R}^2$.

Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\mathbb{R}^2$. Is $A\vec{x} = \vec{b}$ always consistent? The equation has augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$$

. The last column can't be a pivot column, so the system is always consistent.

Hence the linear transformation T is onto.

The Property **One to One**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a **one to one** transformation, then the equation

$$T(\mathbf{x}) = T(\mathbf{y}) \quad \text{is only true when} \quad \mathbf{x} = \mathbf{y}.$$