February 15 Math 2306 sec. 51 Spring 2023

Section 6: Linear Equations Theory and Terminology

We are considering n^{th} order, linear, **homogeneous** equations[1].

$$\boxed{a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0}$$

And we had stated the **principle of superposition** that says that if we have a collection of solutions, $y_1, y_2, \dots, y_k$ to this homogenous ODE, then any function of the form

$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ky_k(x)$$

is also a solution. We called this form a **linear combination**.

[1] Assume a_i are continuous and $a_n(x)\neq 0$ for all x in I.

Linear Dependence/Independence

If we have a set of functions, $f_1(x)$, $f_2(x), \ldots, f_n(x)$, we can form a linear combination that is equal to zero for all x in some interval.

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

Linear Independence

If the **ONLY** way to make this true is for $c_1 = c_2 = \cdots = c_n = 0$ (i.e., all c's must be zero) then the set of functions **Linearly Independent**.

Linear Dependence

If it's possible to make this true with *at least one* of the c's being nonzero, then the set of functions **Linearly Dependent**.

Examples

The set of functions $\{\sin x, \cos x\}$ are linearly **independent** on $(-\infty, \infty)$.

The set of functions $\{x^2, 4x, x - x^2\}$ are linearly **dependent** on $(-\infty, \infty)$.

$$1x^2 - \frac{1}{4}(4x) + 1(x - x^2) = 0$$

I claimed that there would be a *test* that could be used to determine whether a set of functions was linearly dependent or independent. The test involves this thing called a **Wronskian**.

Definition of Wronskian

Definition: Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

We computed this one last time.

$$W(f_1, f_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions $\Rightarrow$ 3×3 matrix.

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1 - 2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x - x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2(-8-0) - 4x\left(-4x - 2(1-2x)\right) + (x-x^2)(0-8)$$

$$= -8x^2 - 4x(-4x - 2 + 4x) - 8x + 8x^2$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

$$W(f_1, f_2, f_3)(x) = 0$$

Theorem (a test for linear independence)

Theorem

Let f_1, $f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

Remark 1: We can compute the Wronskian W as a test:

$$W = 0 \Longrightarrow \text{dependent} \quad \text{or} \quad W \neq 0 \Longrightarrow \text{independent}$$

Remark 2: If the functions $y_1, y_2, \ldots, y_n$ all solve the same linear, homogeneous ODE on some interval I, then their Wronskian is either everywhere zero or nowhere zero on I.

Determine if the functions are linearly dependent or independent:

$$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$$

We can use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= e^x(-2e^{-2x}) - e^x(e^{-2x})$$

$$= -2e^{-x} - e^{-x} = -3e^{-x}$$

$$W(y_1, y_2)(x) = -3e^{-x} \neq 0$$

The functions are linearly independent.

Fundamental Solution Set

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Assume a_i are continuous and $a_n(x) \neq 0$ for all x in I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Assume a_i are continuous and $a_n(x) \neq 0$ for all x in I.

General Solution Homogeneous ODE

Let y_1, $y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

We have to show that we have two, linearly independent solutions.

Let's verify that they are solutions.

$y_1 = x^2$

$y_1' = 2x$

$y_1'' = 2$

$$x^2 y_1'' - 4x y_1' + 6 y_1 \stackrel{?}{=} 0$$

$$x^2(2) - 4x(2x) + 6(x^2) \stackrel{?}{=} 0$$

$$2x^2 - 8x^2 + 6x^2 \stackrel{?}{=} 0$$

$$0 \stackrel{\checkmark}{=} 0$$

y_1 is a solution

$y_2 = x^3$

$y_2' = 3x^2$

$y_2'' = 6x$

$$x^2 y_2'' - 4x\, y_2' + 6 y_2 \stackrel{?}{=} 0$$

$$x^2(6x) - 4x(3x^2) + 6(x^3) \stackrel{?}{=} 0$$

$$6x^3 - 12x^3 + 6x^3 \stackrel{?}{=} 0$$

$$0 \stackrel{\checkmark}{=} 0$$

y_2 is a solution

We have solutions. Let's show that they are linearly independent. Using the Wronskian,

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}.$$

$$= x^2(3x^2) - 2x(x^3)$$

$$= 3x^4 - 2x^4 = x^4$$

$$W(y_1, y_2)(x) = x^4 \neq 0$$

Hence y_1 and y_2 are linearly independent!

We have a fundamental solution set.

The general solution $y = c_1 y_1 + c_2 y_2$

$$\boxed{y = c_1 x^2 + c_2 x^3}$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

General Solution of Nonhomogeneous Equation

General Solution Nonhomogeneous ODE

Let y_p be any solution of the nonhomogeneous equation, and let $y_1, y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

$$y_c = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Superposition Principle (for nonhomogeneous eqns.)

Consider the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g_1(x)+g_2(x) \quad (1)$$

Theorem: If y_{p_1} is a particular solution for

$$a_n(x)\frac{d^ny}{dx^n}+\cdots+a_0(x)y=g_1(x),$$

and y_{p_2} is a particular solution for

$$a_n(x)\frac{d^ny}{dx^n}+\cdots+a_0(x)y=g_2(x),$$

then

$$y_p=y_{p_1}+y_{p_2}$$

is a particular solution for the nonhomogeneous equation (1).

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6 y_{p_1} \stackrel{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$$

$$36 = 36$$

y_{p_1} is a solution

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$

$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$

$28x - 42x \stackrel{?}{=} -14x$

$-14x \stackrel{\checkmark}{=} -14x$

y_{p_2} is a solution

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$y = y_c + y_p$

$y_c = c_1y_1 + c_2y_2$

$y_c = c_1x^2 + c_2x^3$

$y_p = y_{p_1} + y_{p_2}$

$y_p = 6 - 7x$

The general solution is

$$\boxed{y = c_1x^2 + c_2x^3 + 6 - 7x}$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = 5$$

The general solution is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

Find c_1, c_2.

$$y' = 2c_1 x + 3c_2 x^2 - 7$$

$$y(1) = 0 = c_1 (1)^2 + c_2 (1)^3 + 6 - 7(1)$$

$$c_1 + c_2 = 1$$

$$y'(1) = 5 = 2c_1(1) + 3c_2(1)^2 - 7$$

$$2c_1 + 3c_2 = 12$$

$$c_1 + c_2 = 1$$

$$\begin{array}{r} 2c_1 + 3c_2 = 12 \\ -(\,2c_1 + 2c_2 = 2\,) \\ \hline \end{array}$$

$c_2 = 10$

$c_1 = 1 - c_2 = -9$

$$\boxed{\text{The solution to the IVP is } \quad y = -9x^2 + 10x^3 + 6 - 7x}$$