February 16 Math 3260 sec. 52 Spring 2022

Section 1.9: The Matrix for a Linear Transformation

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = \left[T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)\right].$$

The matrix A is called the **standard matrix** for the linear transformation T.

The Property **Onto**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an **onto** transformation, then the equation

$$T(\mathbf{x}) = \mathbf{b}$$

is always solvable. If T is a linear transformation with standard matrix A, then this is equivalent to saying $A\mathbf{x} = \mathbf{b}$ is always consistent.

The Property **One to One**

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a **one to one** transformation, then the equation

$$T(\mathbf{x}) = T(\mathbf{y}) \quad \text{is only true when} \quad \mathbf{x} = \mathbf{y}.$$

Remark: In terms of the standard matrix A, being one-to-one means that

$$\text{if} \quad A\mathbf{x} = \mathbf{b} \quad \text{is consistent, then there is exactly one solution.}$$

Determine if the transformation is one to one.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

Let's call the matrix A. We can consider an equation $A\vec{x} = \vec{b}$ that is consistent. If there is exactly one solution, then T is one to one. If $T: \mathbb{R}^n \to \mathbb{R}^m$, then $n = 3$ and $m = 2$. Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $A\vec{x} = \vec{b}$ has augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}.$$

This is an rref. The solutions satisfy

$$x_1 = b_1 - 2x_3$$
$$x_2 = b_2 - 3x_3 \quad \Longrightarrow \text{ there are infinitely many solutions.}$$
$$x_3 \text{ is free}$$

Hence T is not one to one.

In fact, two different solutions of $T(\vec{x}) = \vec{b}$ are, $\vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} b_1 - 2 \\ b_2 - 3 \\ 1 \end{bmatrix}$

Two Distinct Properties

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

Remark: We saw that this linear transformation

- **IS** onto, but
- it **IS NOT** one to one.

This illustrates that, in general, these two properties are distinct. A transformation could be onto, one-to-one, neither, or both.

Some Theorems on Onto and One to One

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that T is one to one. Is T onto?

Note that the input $\vec{x} = (x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The domain is $\mathbb{R}^2$. The outputs look like (b_1, b_2, b_3). The codomain is $\mathbb{R}^3$.

$T : \mathbb{R}^2 \to \mathbb{R}^3$. Calling the standard matrix A, $A = [T(\vec{e}_1) \quad T(\vec{e}_2)]$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1, 0)$ $\quad T(\vec{e}_1) = T(1,0) = (1, 2\cdot 1 - 0, 3\cdot 0)$

$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$

$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (0,1)$ $\quad T(\vec{e}_2) = T(0,1) = (0, 2\cdot 0 - 1, 3\cdot 1)$

$$= \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

So $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$

To show that T is one to one, we just have to show that $T(\vec{x}) = \vec{0}$ has only the trivial solution, i.e. $A\vec{x} = \vec{0}$ has only the trivial solution.

The augmented matrix for $A\vec{x} = \vec{0}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$$

There are no free variables. $A\vec{x} = \vec{0}$ and hence $T(\vec{x}) = \vec{0}$ have only the trivial solution. This shows that T is one to one.

To determine if T is onto, we can check whether $A\vec{x} = \vec{b}$ is always consistent. For $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, the augmented matrix is $\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & 6b_1 - 2b_2 - b_3 \end{bmatrix}$.

The system is only consistent when $6b_1 - 2b_2 - b_3 = 0$. $T(\vec{x}) = \vec{b}$ is not always consistent, hence T is <u>not</u> onto.