February 19 Math 3260 sec. 51 Spring 2024

Section 2.2: Inverse of a Matrix

Suppose A is an $n \times n$ matrix. We can ask whether there exists another $n \times n$ matrix A^{-1} with the property

$$A^{-1}A = AA^{-1} = I_n.$$

- If such matrix A^{-1} exists, we'll say that A is **nonsingular** or **invertible.**
- Otherwise, we'll say that A is **singular** or **not invertible**..

Theorem

Theorem

Let $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$. If[a] $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right].$$

If $ad - bc = 0$, then A is singular.

[a] The number $ad - bc$ is called the **determinant** of A and can be written as $\det(A)$ or $|A|$.

Theorem

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Inverses, Products, & Transposes

Theorem

(i) If A is invertible, then A^{-1} is also invertible and

$$\left(A^{-1}\right)^{-1} = A.$$

(ii) If A and B are invertible $n \times n$ matrices, then the product AB is also invertible[a] with

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(iii) If A is invertible, then so is A^T. Moreover

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

[a] This can generalize to the product of k invertible matrices.

Elementary Matrices & Row Operations

Definition:

An **elementary** matrix is a square matrix obtained from the identity by performing one elementary row operation.

Remarks

1. Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),
2. Each elementary matrix is invertible where the inverse *undoes* the row operation,
3. Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A.$$

Matrix Inverses

Theorem

An $n \times n$ matrix A is invertible if and only if it is row equivalent to the identity matrix I_n. Moreover, if

$$\text{rref}(A) = E_k \cdots E_2 E_1 A = I_n, \quad \text{then} \quad A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[(E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces A to I_n, transforms I_n into A^{-1}.

Remark: This last observation—operations that take A to I_n also take I_n to A^{-1}—gives us a method for computing an inverse!

Algorithm for finding A^{-1}

Inverse Matrix Algorithm

To find the inverse of a given matrix A:

- Form the $n \times 2n$ augmented matrix $[A \quad I]$.
- Perform whatever row operations are needed to get the first n columns (the A part) to rref.
- If rref(A) is I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$, and the inverse A^{-1} will be the last n columns of the reduced matrix.
- If rref(A) is NOT I, then A is not invertible.

Remarks: We don't need to know ahead of time if A is invertible to use this algorithm. If A is singular, we can stop as soon as it's clear that rref$(A) \neq I$.

Examples: Find the Inverse if Possible

(a) $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix} = A$

Set up $[A \;\; I]$ and do row ops.

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}\right]$$

$4R_1 + R_2 \to R_2$

$2R_1 + R_3 \to R_3$

$$\begin{array}{cccccc} 4 & 8 & -4 & 4 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 2 & 2 & & 1 \end{array}\right]$$

$$\begin{array}{cccccc} 2 & 4 & -2 & 2 & 0 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}$$

$2R_2 + R_3 \to R_3$

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 10 & 2 & 1 \end{array}\right]$$

$0 \quad 2 \quad -2 \quad 8 \quad 2 \quad 0$

$0 \quad -2 \quad 2 \quad 2 \quad 0 \quad 1$

The third column doesn't have a pivot position

$\Rightarrow$ A is singular.

(singular = not invertible)

Examples: Find the Inverse if Possible

(b) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 5 & 6 & 0 \end{bmatrix} = A$

set up $[A \;\; I]$ and do row ops

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 + R_2 \to R_2$

$-5R_1 + R_3 \to R_3$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cccccc} -5 & -10 & -15 & -5 & 0 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}$$

$4R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 4 & 1 \end{bmatrix}$$

$$\begin{matrix} 0 & 4 & 16 & 4 & 4 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{matrix}$$

$-4R_3 + R_2 \to R_2$

$-3R_3 + R_1 \to R_1$

$$\begin{matrix} 0 & 0 & -4 & 4 & -16 & -4 \\ 0 & 1 & 4 & 1 & 1 & 0 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 4 & -12 & -3 \\ 0 & 1 & 0 & 5 & -15 & -4 \\ 0 & 0 & 1 & -1 & 4 & 1 \end{bmatrix}$$

$$\begin{matrix} 0 & 0 & -3 & 3 & -12 & -3 \\ 1 & 2 & 3 & 1 & 0 & 0 \end{matrix}$$

$-2R_2 + R_1 \to R_1$

$$\begin{matrix} 0 & -2 & 0 & -10 & 30 & 8 \\ 1 & 2 & 0 & 4 & -12 & -3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -6 & 18 & 5 \\ 0 & 1 & 0 & 5 & -15 & -4 \\ 0 & 0 & 1 & -1 & 4 & 1 \end{bmatrix}$$

A is nonsingular and

$$A^{-1} = \begin{bmatrix} -6 & 18 & 5 \\ 5 & -15 & -4 \\ -1 & 4 & 1 \end{bmatrix}$$

Try finding A^{-1} by reducing

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \quad -3R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \quad R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{bmatrix} \quad -\frac{1}{2} R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$