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Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Some things to keep in mind:

- Every fundamental solution set has two linearly independent solutions y_1 and y_2,
- The general solution will be

$$y = c_1y_1(x) + c_2y_2(x).$$

Suppose we know one solution $y_1(x)$. This section is about a process called **Reduction of order**. Reduction of order is a method for finding a second solution by assuming that

$$y_2(x) = u(x)y_1(x).$$

The goal is to find the unknown function u.

Context

- We start with a second order, linear, homogeneous ODE in standard form
$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$
- We know one solution y_1. (Keep in mind that y_1 is a known!)
- We know there is a second linearly independent solution (section 6 theory says so).
- We try to find y_2 by guessing that it can be found in the form
$$y_2(x) = u(x)y_1(x)$$
where the goal becomes finding u.
- **Due to linear independence, we know that u cannot be constant.**

Example

Find the general solution to the ODE $x^2y'' - xy' + y = 0$ for $x > 0$ given that $y_1(x) = x$ is one solution.

Assume $y_2 = y_1 u$, i.e. $y_2 = xu$. Sub into the ODE.

$y_2 = xu$

$y_2' = xu' + 1u$

$y_2'' = xu'' + 1u' + 1u'$

$y_2 = xu$

$y_2' = xu' + u$

$y_2'' = xu'' + 2u'$

$x^2y_2'' - xy_2' + y_2 = 0$

$x^2(xu'' + 2u') - x(xu' + u) + xu = 0$

Collect u'', u', and u

$$x^3 u'' + (2x^2 - x^2) u' + (-x + x) u = 0$$

$$x^3 u'' + x^2 u' = 0$$

$$x u'' + u' = 0$$

Let $w = u'$, then $w' = u''$. So w solves

$$x w' + w = 0$$

A 1st order linear and separable ODE.

Let's separate variables

$$x \frac{dw}{dx} = -w$$

$$\frac{1}{w} \frac{dw}{dx} = -\frac{1}{x} \Rightarrow \frac{1}{w} dw = \frac{-1}{x} dx$$

$$\int \frac{1}{w} dw = \int \frac{-1}{x} dx \Rightarrow \ln w = -\ln x$$

assuming $w > 0$.

$$e^{\ln w} = e^{-\ln x} = e^{\ln x^{-1}}$$

$$w = x^{-1}$$

Since $w = u'$, $u = \int w\, dx = \int x^{-1} dx$

$$u = \ln x$$

The 2^{nd} solution

$$y_2 = y_1 u \quad , \quad y_1 = x, \quad u = \ln x$$

so $\quad y_2 = x \ln x$

The general solution is

$$y = c_1 x + c_2 x \ln x$$

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$

Set $\quad y_2 = u y_1 \qquad$ Sub.

$y_2' = u' y_1 + u y_1'$

$y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1''$

$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$

$$\boxed{\text{We know that } \; y_1'' + P(x)y_1' + Q(x)y_1 = 0.}$$

$$y_2'' + P(x)\, y_2' + Q(x)\, y_2 = 0$$

$$u'' y_1 + 2u' y_1' + u y_1'' + P(x)\left(u' y_1 + u y_1'\right) + Q(x)\, u y_1 = 0$$

Collect u'', u', u

$$y_1 u'' + \left(2y_1' + P(x) y_1\right) u' + \underbrace{\left(y_1'' + P(x) y_1' + Q(x) y_1\right)}_{\overset{\parallel}{0}} u = 0$$

Because y_1 is a solution

The equation for u is

$$y_1 u'' + \left(2y_1' + P(x) y_1\right) u' = 0$$

Let $w = u'$, then $w' = u''$. w solves

$$y_1 w' + (2y_1' + P(x) y_1) w = 0$$

This is 1st order linear and separable.

Divide by y_1

$$w' + \left(\frac{2y_1'}{y_1} + P(x) \right) w = 0$$

Separate

$$\frac{dw}{dx} = -\left(2\frac{\frac{dy_1}{dx}}{y_1} + P(x) \right) w$$

$$\frac{1}{w} \frac{dw}{dx} = -\left(2\frac{\frac{dy_1}{dx}}{y_1} + P(x) \right)$$

$$\frac{1}{w}\,dw = -\left(2\frac{\frac{dy_1}{dx}}{y_1} + P(x)\right)dx$$

$$\frac{1}{w}\,dw = -2\frac{dy_1}{y_1} - P(x)\,dx$$

$$\int \frac{1}{w}\,dw = -2\int \frac{dy_1}{y_1} - \int P(x)\,dx$$

$$\ln w = -2\ln|y_1| - \int P(x)\,dx$$

$$\ln w = \ln y_1^{-2} - \int P(x)\,dx$$

$$e^{\ln w} = e^{\ln y_1^{-2} - \int P(x)\,dx}$$

$$w = e^{\ln y_1^{-2}} \cdot e^{-\int p(x)\,dx}$$

$$w = y_1^{-2}\, e^{-\int p(x)\,dx}$$

$$w = \frac{e^{-\int p(x)\,dx}}{y_1^2}$$

$$w = u' \Rightarrow u = \int w\,dx$$

$$\boxed{u = \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx}$$

Then $\quad y_2 = y_1 u$

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2(x) = y_1(x)u(x) \quad \text{where} \quad u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Gen soln. $y = c_1 y_1 + c_2 y_2$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$

$$u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Find y_2: $\quad P(x) = 4$

$$-\int P(x)\,dx = -\int 4\,dx = -4x$$

$$u = \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx = \int \frac{e^{-4x}}{(e^{-2x})^2}\,dx$$

$$= \int \frac{e^{-4x}}{e^{-4x}}\,dx = \int dx = x$$

So $u = x$ and $y_2 = u y_1 = x e^{-2x}$

The general solution $y = c_1 y_1 + c_2 y_2$

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

Apply $y(0) = 1$ and $y'(0) = 1$

$$y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$$

$$y(0) = c_1 e^{0} + c_2 \cdot 0 \cdot e^{0} = 1 \Rightarrow c_1 = 1$$

$$y'(0) = -2c_1 e^{0} + c_2 e^{0} - 2c_2 \cdot 0 \cdot e^{0} = 1$$

$$-2c_1 + c_2 = 1$$

$$c_2 = 1 + 2c_1 = 1 + 2(1) = 3$$

The solution to the IVP is

$$\boxed{y = e^{-2x} + 3x\,e^{-2x}}$$