February 20 Math 2306 sec. 51 Spring 2023

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

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where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Some things to keep in mind:

- Every fundamental solution set has two linearly independent solutions y₁ and y₂,
- The general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we know one solution $y_1(x)$. This section is about a process called **Reduction of order**. Reduction of order is a method for finding a second solution by assuming that

 $y_2(x) = u(x)y_1(x).$

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The goal is to find the unknown function *u*.

Context

We start with a second order, linear, homogeneous ODE in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

- We know one solution y_1 . (Keep in mind that y_1 is a known!)
- We know there is a second linearly independent solution (section 6 theory says so).
- We try to find y_2 by guessing that it can be found in the form

$$y_2(x) = u(x)y_1(x)$$

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where the goal becomes finding *u*.

Due to linear independence, we know that u cannot be constant.

Example

Find the general solution to the ODE $x^2y'' - xy' + y = 0$ for x > 0 given that $y_1(x) = x$ is one solution.

Assume
$$y_{2} = y_{1}u_{1}$$
, i.e. $y_{2} = \chi u_{1}$. Sub into
the ODC.
 $y_{2} = \chi u_{1}$
 $y_{2}' = \chi u'_{1} + 1u_{1}$
 $y_{2}'' = \chi u''_{1} + 1u'_{1} + 1u'_{1}$
 $\chi^{2}y_{2}''_{1} - \chi y_{2}' + y_{2} = 0$
 $\chi^{2}(\chi u''_{1} + 2u') - \chi (\chi u'_{1} + u) + \chi u_{1} = 0$
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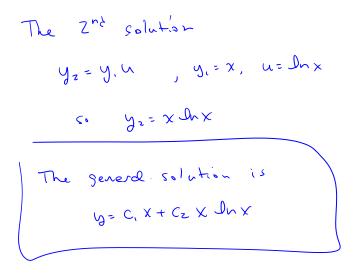
Collect
$$u'', u', m \in C$$

 $X^{3}u'' + (2x^{2} - x^{2})u' + (-x + x)u = 0$
 $x^{3}u'' + x^{2}u' = 0$
 $xu'' + u' = 0$
Let $v = u'$, then $w' = u''$. So w solves
 $xw' + w = 0$
A 1st order linear and separable ODE.
Let's separate variables
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$$\begin{array}{l} x \frac{dw}{dx} = -w \\ \frac{1}{w} \frac{dw}{dx} = -\frac{1}{x} \implies \frac{1}{w} \frac{dw}{dw} = \frac{1}{x} \frac{dx}{dx} \\ \int \frac{1}{w} \frac{dw}{dx} = -\frac{1}{x} \implies \frac{1}{w} \frac{dw}{dw} = \frac{1}{x} \frac{dx}{dx} \\ \int \frac{1}{w} \frac{dw}{dw} = \int \frac{1}{x} \frac{dx}{dx} \implies \frac{1}{w} \frac{w}{w} = -\frac{1}{w} \frac{1}{x} \\ assuming w = 0 \\ e^{nw} = e^{nx} = e^{nx} \\ e^{nw} = e^{nx} = e^{nx} \\ w = x^{2} \\ since w = w' \\ u = \int w \frac{1}{w} \frac{1}{w}$$

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Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$
Set
$$y_z = Uy_1, \quad \text{Sub}.$$

$$y_z' = u'y_1 + uy_1'$$

$$y_z'' = u''y_1 + uy_1' + uy_1' + uy_1''$$

$$y_z'' = u''y_1 + 2U'y_1' + uy_1''$$
We know that $y_1'' + P(x)y_1' + Q(x)y_1 = 0.$

$$y_{x}'' + P(x) y_{x}' + Q(x) y_{z} = 0$$

$$u''y_{1} + zu'y_{1}' + wy_{1}'' + P(x) (u'y_{1} + uy_{1}') + Q(x) uy_{1} = 0$$
Gilled u'', u', u''
$$y_{1} u'' + (zy_{1}' + P(x)y_{1}) u' + (y_{1}'' + P(x)y_{1}' + Q(x)y_{1}) u = 0$$

$$y_{1} u'' + (zy_{1}' + P(x)y_{1}) u' + (y_{1}'' + P(x)y_{1}' + Q(x)y_{1}) u = 0$$

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$$y_{3} u'' + (zy_{1}' + P(x)y_{1}) u' = 0$$

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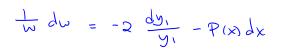
het W = u', then w' = u''. w solves $y_1w' + (2y_1' + P(x)y_1)w = 0$ This is 1st order linear and separable. Divide by y_1

$$W' + \left(\frac{2y'_{i}}{y_{i}} + P(x)\right)W = 0$$

Separate $\frac{d\omega}{dx} = -\left(2\frac{dy_1}{y_1} + P(x)\right)W$ $\frac{1}{w}\frac{dw}{dx} = -\left(2\frac{dy_1}{y_1} + P(x)\right)$

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$$\frac{1}{2} dw = -\left(2 \frac{dy_1}{dx} + P(x)\right) dx$$



$$\int \frac{dw}{dw} = -2 \int \frac{dy_1}{y_1} - \int P(x) dx$$

 $\ln \omega = \ln y_i^2 - \int f(x) \, dx$

 $e^{\int w} = e^{\int y_i^2 - \int g(x) dx}$

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$$W = e^{\int_{0}^{1} y_{1}^{2}} - \int P(x) dx$$
$$W = y_{1}^{2} e^{-\int P(x) dx}$$
$$W = \frac{e^{-\int P(x) dx}}{y_{1}^{2}}$$

$$u = \int \frac{-\int \rho(x) dx}{g_{1}^{2}} dx$$

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Then $y_z = y_1 u$

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 $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

Reduction of Order Formula

For the second order, homogeneous equation in standard form with one known solution y_1 , a second linearly independent solution y_2 is given by

$$y_2(x) = y_1(x)u(x)$$
 where $u(x) = \int \frac{e^{-\int P(x) \, dx}}{(y_1(x))^2} \, dx$

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Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$

Find y_2 : $P(x) = Y$

$$u(x) = \int \frac{e^{-\int P(x) dx}}{(y_1(x))^2} dx$$

$$-\int P(x) dx = -\int Y dx = -4x$$

$$u = \int \frac{e^{\int P(x) dx}}{(y_1(x))^2} dx$$

$$= \int \frac{e^{\int Y}}{(y_1)^2} dx = \int \frac{e^{\int Y}}{(e^{2x})^2} dx$$

$$= \int \frac{e^{\int Y}}{(e^{2x})^2} dx = \int dx = x$$

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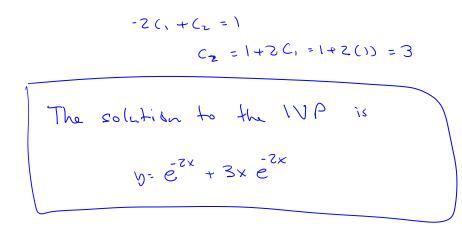
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$$Y = C_1 e^{-2x} + C_2 \times e^{-2x}$$

Apply
$$y(0) = 1$$
 and $y'(0) = 1$
 $y' = -zc_1 e^{2x} + c_2 e^{2x} - zc_2 e^{2x}$
 $y(0) = c_1 e^{0} + c_2 \cdot 0 \cdot e^{0} = 1 \Rightarrow c_1 = 1$
 $y'(0) = -2c_1 e^{0} + c_2 e^{0} - 2c_2 \cdot 0 \cdot e^{0} = 1$

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