February 21 Math 3260 sec. 51 Spring 2022

Section 2.1: Matrix Operations

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar c

$$cA = [ca_{ij}].$$

Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}].$$

Matrix Multiplication: If A is $m \times n$ and B is $n \times p$, then the product AB is defined by

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p].$$

The product AB is $m \times p$. Moreover, if $AB = C = [c_{ij}]$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Product of Matrices

The product AB is only defined if the number of columns of A (the left matrix) matches the number of rows of B (the right matrix).

$$A\ B$$

$$m \times n\ \ n \times p$$

must match

$$\downarrow$$

$$m \times p$$

product dimensions

Properties of Scalar Multiplication and Matrix Addition

Theorem: Let A, B, and C be matrices of the same size and r and s be scalars. Then

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + O = A$

(iv) $r(A + B) = rA + rB$

(v) $(r + s)A = rA + sA$

(vi) $r(sA) = (rs)A$

where O is the zero matrix of the same size as A.

Properties of the Product of Matrices

Theorem: Let A be an $m \times n$ matrix. Let r be a scalar and B and C be matrices for which the indicated sums and products are defined. Then

(i) $A(BC) = (AB)C$

The order for each product can't be changed.

(ii) $A(B + C) = AB + AC$

(iii) $(B + C)A = BA + CA$

(iv) $r(AB) = (rA)B = A(rB)$, and

(v) $I_m A = A = A I_n$

Caveats!

(1) Matrix multiplication **does not** commute! In general $AB \neq BA$

(2) The zero product property **does not** hold! That is, if $AB = O$, one **cannot** conclude that one of the matrices A or B is a zero matrix.

(3) There is no *cancelation law*. That is, $AB = CB$ **does not** imply that A and C are equal.

Compute AB and BA where $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$.

$$\underbrace{A \quad B}_{}$$
$$\underbrace{2\times 2 \quad 2\times 2}_{\text{match}}$$

$$\underbrace{B \quad A}_{}$$
$$\underbrace{2\times 2 \quad 2\times 2}_{\text{match}}$$

row 1 column 1
row 1 from A
Column 1 from B
dot product

$1 \cdot 4 + 2(-1)$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ -1 & 4 \end{bmatrix}$$

Both products are defined, but $AB \neq BA$.

Compute the products AB, CB, and BB where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

All are defined and all are 2x2.

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

Note $AB = CB$ but $A \neq C$

$$CB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BB = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note $BB = 0$ but $B \neq 0$

Matrix Powers

Positive Integer Powers: If A is square—meaning A is an $n \times n$ matrix for some $n \geq 2$, then the product AA is defined. For positive integer k, we'll define

$$A^k = AA^{k-1}.$$

$A^2 = AA$

$A^3 = AA^2$

$A^4 = AA^3$

Zero Power: We define $A^0 = I_n$, where I_n is the $n \times n$ identity matrix.

Transpose

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix denoted and defined by

$$A^T = [a_{ji}].$$

For example, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Compute A^T, B^T, the transpose of the product $(AB)^T$, and the product B^TA^T.

We already computed $AB = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$ in a previous example.

$$A^T = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -1 & -2 \\ 12 & -8 \\ -16 & 8 \end{bmatrix}.$$

$$\underbrace{\underset{3\times 2}{B^T}\ \underset{2\times 2}{A^T}}_{\text{product is } 3\times 2}$$

match

$$B^T A^T = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 12 & -8 \\ -16 & 8 \end{bmatrix}$$

Theorem: Properties-Matrix Transposition

Let A and B be matrices such that the appropriate sums and products are defined, and let r be a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$

$(ABC)^T = C^T B^T A^T$ etc.

Section 2.2: Inverse of a Matrix

Consider the scalar equation $ax = b$. Provided $a \neq 0$, we can solve this explicity
$$x = a^{-1}b$$
where a^{-1} is the unique number such that $aa^{-1} = a^{-1}a = 1$.

If A is an $n \times n$ matrix, we seek an analog A^{-1} that satisfies the condition
$$A^{-1}A = AA^{-1} = I_n.$$

If such matrix A^{-1} exists, we'll say that A is **nonsingular** (a.k.a. *invertible*). Otherwise, we'll say that A is **singular**.

Theorem (2×2 case)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is singular.

The quantity $ad - bc$ is called the **determinant** of A and may be denoted in several ways

$$\det(A) \quad = \quad |A| \quad = \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Find the inverse if possible

(a) $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$

$A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

$\det(A) = 3(5) - (-1)(2) = 17 \neq 0$

A^{-1} exists

(b) $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

A is singular, A^{-1} doesn't exist

$\det(A) = 3 \cdot 4 - 6 \cdot 2 = 0$