February 21 Math 3260 sec. 52 Spring 2022

Section 2.1: Matrix Operations

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar c $cA = [ca_{ii}]$.

Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ $A + B = [a_{ii} + b_{ii}]$.

Matrix Multiplication: If *A* is $m \times n$ and *B* is $n \times p$, then the product *AB* is defined by

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p].$$

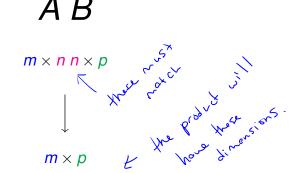
The product AB is $m \times p$. Moreover, if $AB = C = [c_{ij}]$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$



Product of Matrices

The product AB is only defined if the number of columns of A (the left matrix) matches the number of rows of B (the right matrix).



Properties of Scalar Multiplication and Matrix Addition

Theorem: Let *A*, *B*, and *C* be matrices of the same size and *r* and *s* be scalars. Then

(i)
$$A + B = B + A$$
 (iv) $r(A + B) = rA + rB$

(ii)
$$(A + B) + C = A + (B + C)$$
 (v) $(r + s)A = rA + sA$

(iii)
$$A + O = A$$
 (vi) $r(sA) = (rs)A$

where O is the zero matrix of the same size as A.

Properties of the Product of Matrices

Theorem: Let A be an $m \times n$ matrix. Let r be a scalar and B and C be matrices for which the indicated sums and products are defined. Then

(i)
$$A(BC) = (AB)C$$

(ii)
$$A(B+C) = AB + AC$$

(iii)
$$(B+C)A = BA + CA$$

The order

in these

products be

contained

(iv)
$$r(AB) = (rA)B = A(rB)$$
, and

(v)
$$I_m A = A = A I_n$$

Caveats!

(1) Matrix multiplication **does not** commute! In general $AB \neq BA$

(2) The zero product property **does not** hold! That is, if AB = O, one **cannot** conclude that one of the matrices A or B is a zero matrix.

(3) There is no *cancelation law*. That is, AB = CB does not imply that A and C are equal.

Compute
$$AB$$
 and BA where $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 6 \end{bmatrix}$$

$$Coult from A$$

$$dol vill$$

$$Column 1 of ($$

$$BA = \begin{bmatrix} u & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ -1 & 4 \end{bmatrix}$$

Both products are defined, but AB = BA.

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Column 1 of B

· (.(4)+2(-1)=Z

Compute the products *AB*, *CB*, and *BB* where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$$B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$$
, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

All products are defined and will be 2x2.

$$AB : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$CB : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Note
$$AB = CB$$

but $A \neq C$

Matrix Powers

$$A^{k} = AA^{k-1}.$$

$$A^{3} = AA^{2}$$

$$A^{4} = AA^{3}$$

Zero Power: We define $A^0 = I_n$, where I_n is the $n \times n$ identity matrix.

Transpose

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix denoted and defined by

$$A^T = [a_{jj}].$$

For example, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$
, then $A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$.

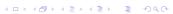
Example

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Compute A^T , B^T , the transpose of the product $(AB)^T$, and the product B^TA^T .

We already computed $AB = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$ in a previous example.

$$A^{T} = \begin{bmatrix} 1 & -2 \\ -3 & z \end{bmatrix} , \quad B^{T} = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix}$$



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$$(AB)^{T} = \begin{bmatrix} -1 & -2 \\ 12 & -8 \\ -16 & 8 \end{bmatrix}$$

$$3 \times 2 \quad 2 \times 2$$

$$product \quad 15 \quad 3 \times 2$$

Theorem: Properties-Matrix Transposition

Let *A* and *B* be matrices such that the appropriate sums and products are defined, and let *r* be a scalar. Then

(i)
$$(A^T)^T = A$$

(ii)
$$(A + B)^T = A^T + B^T$$

(iii)
$$(rA)^T = rA^T$$

(iv)
$$(AB)^T = B^T A^T$$

Section 2.2: Inverse of a Matrix

Consider the scalar equation ax = b. Provided $a \neq 0$, we can solve this explicity

$$x = a^{-1}b$$

where a^{-1} is the unique number such that $aa^{-1} = a^{-1}a = 1$.

If A is an $n \times n$ matrix, we seek an analog A^{-1} that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n$$
.

If such matrix A^{-1} exists, we'll say that A is **nonsingular** (a.k.a. *invertible*). Otherwise, we'll say that A is **singular**.

Theorem (2 \times 2 case)

Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If ad - bc = 0, then A is singular.

The quantity ad - bc is called the **determinant** of A and may be denoted in several ways

$$det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$



Find the inverse if possible

(a)
$$A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{17} \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$$

