## February 22 Math 2306 sec. 51 Spring 2023

#### Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order<sup>1</sup>, linear, homogeneous equation with constant coefficients

$$arac{d^2y}{dx^2}+brac{dy}{dx}+cy=0, \quad ext{with } a
eq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\frac{dy}{dx} - \frac{c}{a}y.$$

Question: What sorts of functions y could be expected to satisfy

$$y'' = (\text{constant}) y' + (\text{constant}) y?$$
  
Exponentials  $y = e^{x}$ ,  $y = sm(kx)$  or  $y = cos(kx)$   
 $m$ ,  $k - constants$ 

<sup>1</sup>We'll extend the result to higher order at the end of this section. February 20, 2023 1/25

# We look for solutions of the form $y = e^{mx}$ with *m* constant.

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0,$$

$$y = e^{mx}$$

$$ay'' + by' + cy = 0$$

$$y' = me^{mx}$$

$$a(m^2 e^{mx}) + b(me^{mx}) + c(e^{mx}) = 0$$

$$y'' = m^2 e^{mx}$$

$$e^{mx}(am^2 + bm + c) = 0$$
This will be true for all x  
in some interval if

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 $am^2 + bm + c = 0$ 

If m solves this guadratic equation, then y= ex solver the ODE.

#### Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

- $| b^2 4ac > 0$  and there are two distinct real roots  $m_1 \neq m_2$
- II  $b^2 4ac = 0$  and there is one repeated real root  $m_1 = m_2 = m_1$
- III  $b^2 4ac < 0$  and there are two roots that are complex conjugates  $m_1 = \alpha + i\beta$  and  $m_2 = \alpha - i\beta$ .

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#### Case I: Two distinct real roots

$$ay'' + by' + cy = 0$$
, where  $b^2 - 4ac > 0$ .

There are two different roots  $m_1$  and  $m_2$ . A fundamental solution set consists of

$$y_1 = e^{m_1 x}$$
 and  $y_2 = e^{m_2 x}$ .

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

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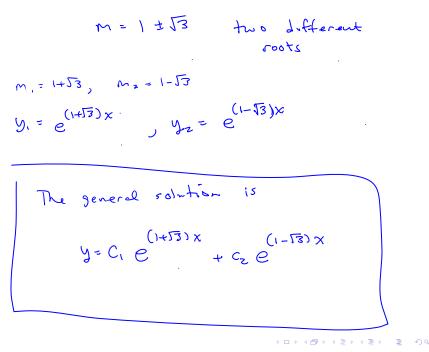
### Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$
  
The characteristic equation is  
$$m^{2} - 2m - 7 = 0$$
  
Completing the square.  
$$m^{2} - 2m + 1 - 1 - 2 = 0$$
  
$$(m - 1)^{2} - 3 = 0 = 3 \quad (m - 1)^{2} = 3$$
  
$$m - 1 = \pm \sqrt{3}$$

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Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where  $b^2 - 4ac = 0$ 

There is only one real, double root,  $m = \frac{-b}{2a}$ .

Use reduction of order to find the second solution to the equation (in standard form)

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$$

$$\mathbb{P}(x) = \frac{b}{a}, \quad -\int \mathbb{P}(x)dx = -\int \frac{b}{a}dx = -\frac{b}{a}x$$

$$u = \int \frac{-\int \mathbb{P}(x)dx}{\left(\frac{e}{y_1}\right)^2}dx = \int \frac{e^{-\frac{b}{a}x}}{\left(\frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\right)^2}dx$$

$$= \int \frac{e^{-\frac{b}{a}x}}{\left(\frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\right)^2}dx$$

 $\left(e^{-\frac{b}{2\alpha}\chi}\right)^2 = e^{\left(-\frac{b}{2\alpha}\chi\right)^2} = e^{-\frac{b}{\alpha}\chi}$ 

 $u = \int \frac{\frac{-b}{dx}}{\frac{-b}{dx}} dx = \int 1 dx = x$ ~<u>b</u> y,= e  $y_z = uy,$  $y_z = x e^{-\frac{b}{2x}x}$ 

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#### Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where  $b^2 - 4ac = 0$ 

If the characteristic equation has one real repeated root *m*, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx}$$
 and  $y_2 = xe^{mx}$ .

The general solution is

$$y=c_1e^{mx}+c_2xe^{mx}.$$

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