February 22 Math 2306 sec. 52 Spring 2023

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, \quad \text{with } a\neq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2}=-\frac{b}{a}\frac{dy}{dx}-\frac{c}{a}y.$$

Question: What sorts of functions y could be expected to satisfy

$$y''=(\text{constant})\,y'+(\text{constant})\,y?$$

$y=e^{mx}$, m - constant $\qquad y=\sin(kx)$, or $y=\cos(kx)$

k - constant

[1] We'll extend the result to higher order at the end of this section.

We look for solutions of the form $y = e^{mx}$ with m constant.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

$y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

$ay'' + by' + cy = 0$

$a(m^2e^{mx}) + b(me^{mx}) + c(e^{mx}) = 0$

$e^{mx}(am^2 + bm + c) = 0$

This will hold if m solves the quadratic equation

$$am^2 + bm + c = 0$$

For such m, $y = e^{mx}$ will solve

the ODE

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$

The characteristic equation is

$$m^2 - 2m - 2 = 0$$

Completing the square

$$m^2 - 2m + 1 - 1 - 2 = 0$$

$$(m-1)^2 - 3 = 0 \implies (m-1)^2 = 3$$

$$m - 1 = \pm\sqrt{3}$$

$m = 1 \pm \sqrt{3}$ two distinct real roots,

$m_1 = 1+\sqrt{3}$, $m_2 = 1-\sqrt{3}$

$y_1 = e^{(1+\sqrt{3})x}$, $y_2 = e^{(1-\sqrt{3})x}$

The general solution is

$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

There is only one real, double root, $m = \dfrac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$$

$y_2 = u y_1$ where $u = \displaystyle\int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

$P(x) = \frac{b}{a}$, $-\int P(x)\,dx = \int -\frac{b}{a}\,dx = -\frac{b}{a}x$

$$(y_1)^2 = \left(e^{-\frac{b}{2a}x} \right)^2 = e^{-2\left(\frac{b}{2a}x\right)} = e^{-\frac{b}{a}x}$$

$$u = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\, dx = \int 1\, dx = x$$

$$y_2 = u y_1 \quad , \quad y_1 = e^{-\frac{b}{2a}x}$$

$$y_2 = x\, e^{-\frac{b}{2a}x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$