February 23 Math 3260 sec. 51 Spring 2022

Section 2.2: Inverse of a Matrix

Question: Given an $n \times n$ matrix A, is there a matrix A^{-1} that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n.$$

If such matrix A^{-1} exists, we'll say that A is **nonsingular** (a.k.a. *invertible*). Otherwise, we'll say that A is **singular**.

Theorem (2×2 case)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is singular.

The quantity $ad - bc$ is called the **determinant** of A and may be denoted in several ways

$$\det(A) \quad = \quad |A| \quad = \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Find the inverse if possible

(a) $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$

We found that A is nonsingular with inverse $A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

This matrix turned out to be singular. Its determinant $\det(A) = 0$.

Theorem

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

To prove this, we have to show that

① $A^{-1}\vec{b}$ is a solution, and

② If $\vec{y}$ is a solution then $\vec{y} = A^{-1}\vec{b}$.

Suppose A is invertible and let $\vec{x} = A^{-1}\vec{b}$.

Subbing into the matrix equation

$$A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I_n\vec{b} = \vec{b}$$

Hence $A^{-1}\vec{b}$ is a solution to $A\vec{x} = \vec{b}$.

For the second part, suppose $\vec{y}$ solves $A\vec{x} = \vec{b}$. Then

$$A\vec{y} = \vec{b}.$$

Multiply each side of this equation on the left by A^{-1}.

$$A^{-1}A\vec{y} = A^{-1}\vec{b} \implies (A^{-1}A)\vec{y} = A^{-1}\vec{b}$$

$$\implies I_n\vec{y} = A^{-1}\vec{b}$$

$$\implies \vec{y} = A^{-1}\vec{b}$$

So $A^{-1}\vec{b}$ is a unique solution to $A\vec{x} = \vec{b}$.

Example

Solve the system

$$\begin{aligned} 3x_1 &+ 2x_2 &= -1 \\ -x_1 &+ 5x_2 &= 4 \end{aligned}$$

Let's use a matrix inverse

$$\Longrightarrow \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Call this $A\vec{x} = \vec{b}$

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$ from previous example.

The solution $\vec{x} = A^{-1}\vec{b}$

$$\vec{x} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -13 \\ 11 \end{bmatrix} = \begin{bmatrix} -13/17 \\ 11/17 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= \frac{-13}{17} \\ x_2 &= \frac{11}{17}. \end{aligned}$$

Theorem

(i) If A is invertible, then A^{-1} is also invertible and

$$\left(A^{-1}\right)^{-1} = A.$$

(ii) If A and B are invertible $n \times n$ matrices, then the product AB is also invertible[1] with

$$(AB)^{-1} = B^{-1}A^{-1}.$$

$(AB)^{-1}AB = I$

(iii) If A is invertible, then so is A^T. Moreover

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

$AA^{-1} = I$

$(AA^{-1})^T = I^T$

[1] This can generalize to the product of k invertible matrices.

Elementary Matrices

Definition: An **elementary** matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$3R_2 \to R_2$ $\qquad$ $2R_1 + R_3 \to R_3$ $\qquad$ $R_1 \leftrightarrow R_2$

Action of Elementary Matrices

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and compute the following products

$$E_1A, \quad E_2A, \quad \text{and} \quad E_3A.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$$

3×3 3×3

match

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 3R_2 \to R_2$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 2a+g & 2b+h & 2c+i \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$2R_1 + R_3 \to R_3$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}.$$

$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

Remarks

- Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),
- Each elementary matrix is invertible where the inverse *undoes* the row operation,
- Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A.$$

Theorem

An $n \times n$ matrix A is invertible if and only if it is row equivalent to the identity matrix I_n. Moreover, if

$$\text{rref}(A) = E_k \cdots E_2 E_1 A = I_n, \quad \text{then} \quad A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[(E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces A to I_n, transforms I_n into A^{-1}.

This last observation—operations that take A to I_n also take I_n to A^{-1}—gives us a method for computing an inverse!

Algorithm for finding A^{-1}

To find the inverse of a given matrix A:

- Form the $n \times 2n$ augmented matrix $[A \quad I]$.
- Perform whatever row operations are needed to get the first n columns (the A part) to rref.
- If rref(A) is I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$, and the inverse A^{-1} will be the last n columns of the reduced matrix.
- If rref(A) is NOT I, then A is not invertible.

Remarks: We don't need to know ahead of time if A is invertible to use this algorithm.
If A is singular, we can stop as soon as it's clear that rref$(A) \neq I$.

Examples: Find the Inverse if Possible

$$A$$

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

Set up augmented matrix $[A \quad I]$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} -5 & -10 & -15 & -5 & 0 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{matrix}$$

$-5R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{bmatrix}$$

$4R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$$\begin{array}{cccccc} 0 & 4 & 16 & 0 & 4 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}$$

$-4R_3 + R_2 \to R_2$

$-3R_3 + R_1 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$$\begin{array}{cccccc} 0 & 0 & -4 & 20 & -16 & -4 \\ 0 & 1 & 4 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{cccccc} 0 & 0 & -3 & 15 & -12 & -3 \\ 1 & 2 & 3 & 1 & 0 & 0 \end{array}$$

$-2R_2 + R_1 \to R_1$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$$\begin{array}{cccccc} 0 & -2 & 0 & -40 & 30 & 8 \\ 1 & 2 & 0 & 16 & -12 & -3 \end{array}$$

$$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$