February 23 Math 3260 sec. 51 Spring 2024

Section 2.3: Characterization of Invertible Matrices

Given an $n \times n$ matrix A, we can think of...

- A matrix equation $A\mathbf{x} = \mathbf{b}$;
- A linear system that has A as its coefficient matrix;
- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$;
- Not to mention things like its **pivots**, its **rref**, the linear dependence/independence of its columns, blah blah blah...

Question: Is this stuff related to being singular/nonsingular? How?

The Invertible Matrix Theorem

Suppose A is $n \times n$. The following are equivalent. [a]

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.

(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(j) There exists an $n \times n$ matrix C such that $CA = I$.

(k) There exists an $n \times n$ matrix D such that $AD = I$.

(l) A^T is invertible.

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[a] Meaning all are true or none are true.

The Inverse of a Matrix is Unique

Theorem

Let A and B be $n \times n$ matrices. If $AB = I$, then A and B are both invertible with $A^{-1} = B$ and $B^{-1} = A$.

Proof: Suppose $AB = I$. Let's show that B is invertible and $B^{-1} = A$. Consider the homogeneous system $B\vec{x} = \vec{0}$. Multiply on the left by A.

$$B\vec{x} = \vec{0}$$
$$AB\vec{x} = A\vec{0}$$
$$I\vec{x} = \vec{0}$$
$$\vec{x} = \vec{0} \quad \Rightarrow B\vec{x} = \vec{0} \text{ has only}$$

the trivial solution. It follows that B is invertible ((d) ⇒ (a)), that is B^{-1} exists. From $AB = I$, multiply on the right by B^{-1}.

$$AB = I$$
$$ABB^{-1} = IB^{-1}$$
$$AI = B^{-1}$$
$$A = B^{-1}$$

Since B^{-1} is also invertible, A is invertible. And

$$A = B^{-1}$$
$$A^{-1} = (B^{-1})^{-1}$$

$A^{-1} = B$.

Invertible Linear Transformations

Definition:

A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that both

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad T(S(\mathbf{x})) = \mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

If such a function exists, we typically denote it by

$$S = T^{-1}.$$

Invertability of a Transformation and its Matrix

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and A its standard matrix. Then T is invertible if and only if A is invertible. Moreover, if T is invertible, then

$$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

Remark: This indicates that we can determine if a linear transformation is invertible and identify the inverse transform using the standard matrix.

Example

Use the standard matrix to determine if the linear transformation is invertible. If it is invertible, characterize the inverse transformation.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \text{given by} \quad T(x_1, x_2) = (3x_1 - x_2, 4x_2).$$

Calling the standard matrix A, $A = [T(\vec{e}_1) \ T(\vec{e}_2)]$

$T(\vec{e}_1) = T(1,0) = (3\cdot 1 - 0, 4\cdot 0) = (3,0)$

$T(\vec{e}_2) = T(0,1) = (3\cdot 0 - 1, 4\cdot 1) = (-1,4)$

$A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$. T^{-1} exists if A^{-1} exists

$\det(A) = 3\cdot 4 - 0(-1) = 12 \neq 0$ A^{-1} exists.

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}.$$

we can find $T^{-1}(\vec{x}) = A^{-1}\vec{x}$.

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{12}\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 4x_1 + x_2 \\ 0x_1 + 3x_2 \end{bmatrix}.$$

$$= \begin{bmatrix} \frac{4}{12}x_1 + \frac{1}{12}x_2 \\ \frac{3}{12}x_2 \end{bmatrix}$$

$$\boxed{T^{-1}(x_1, x_2) = \left(\frac{1}{3}x_1 + \frac{1}{12}x_2,\ \frac{1}{4}x_2\right)}$$

Example

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a one to one linear transformation. Can we determine whether T is onto? Why (or why not)?

Yes, T is onto. It's standard matrix, A would be invertible (since $\vec{x} \mapsto A\vec{x}$ being one to one $\Rightarrow$ A is invertible.) This implies that $\vec{x} \mapsto A\vec{x}$ is onto — i.e., T is onto.

Section 3.1: Introduction to Determinants

We defined a number, called a **determinant**, for a 2×2 matrix. And that number was related to whether the matrix was invertible.

For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we said that the determinant

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}.$$

And we had the critical relationship that A is nonsingular (a.k.a. invertible) if and only if $\det(A)$ is nonzero.

Here, we want to extend the concept of **determinant** to all $n \times n$ matrices and do it in such a way that for any square matrix A,

A is nonsinguar if and only if $\det(A) \neq 0$.

Determinant: 3×3 Matrix

Let's assume that $A = [a_{ij}]$ is an **invertible** 3×3 matrix, and suppose that $a_{11} \neq 0$. We can start the row reduction process to obtain zeros below the left most pivot position.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{array}{c} a_{11}R_2 \to R_2 \\ a_{11}R_3 \to R_3 \end{array} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

$$\begin{array}{c} -a_{21}R_1 + R_2 \to R_2 \\ -a_{31}R_1 + R_3 \to R_3 \end{array} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Determinant: 3×3 Matrix

If $A \sim I$, one of the entries in the $2,2$ or the $3,2$ position must be nonzero. Let's assume it is the $2,2$ entry and continue the reduction[1]

$$\begin{array}{r} b_{22}R_3 \to R_3 \\ -b_{32}R_2 + R_3 \to R_3 \end{array} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix},$$

where Δ is an expression involving the entries of A. We can state the following:

> If A is invertible, it must be that the bottom right entry is nonzero. That is
>
> $$\Delta \neq 0.$$
>
> Note that if $\Delta = 0$, then A would not be row equivalent to I making A singular. We will define the determinant to be Δ.

[1] The factors shown here are $b_{22} = a_{11}a_{22} - a_{12}a_{21}$ and $b_{32} = a_{11}a_{32} - a_{12}a_{31}$.

Determinant: 3×3 Matrix

We can rearrange the term Δ and state the determinant in an easy to remember way.

$$\Delta = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Notice that each expression in parentheses is a *product minus product*, i.e., they look like determinants of 2×2 matrices! We can restate these as determinants and arrive at the following formula for the determinant of a 3×3 matrix.

3×3 Determinant

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$