February 24 Math 2306 sec. 51 Spring 2023

Section 8: Homogeneous Equations with Constant Coefficients

We were considering second order, linear, homogeneous ODEs with constant coefficients.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \quad \text{with } a \neq 0.$$

We looked for solutions of the form $y = e^{mx}$ for constant m and found that we'll get such solutions is m is a root of the
characteristic equation

$$am^2 + bm + c = 0.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

We talked about the first two and need to consider the last one.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set[1] to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

[1] See workbook exercise 2(b) from sec. 6 on linear independence.

Example

Solve the IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

Solve the ODE.

Characteristic eqn.

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0 \implies m = -3 \text{ double root}$$

$$y_1 = e^{-3x}, \quad y_2 = x e^{-3x}$$

The general solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x}$$

Apply $y(0) = 4$, $y'(0) = 0$

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = 4 \Rightarrow c_1 = 4$$

$$y'(0) = -3c_1 e^0 + c_2 e^0 - 3c_2 \cdot 0 \cdot e^0 = 0$$

$$-3c_1 + c_2 = 0$$

$$c_2 = 3c_1 = 3(4) = 12$$

$$y = c_1 e^{-3x} + c_2 x e^{-3x}, \quad c_1 = 4, \quad c_2 = 12$$

The solution to the IVP is

$$y = 4e^{-3x} + 12xe^{-3x}$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

The two roots of the characteristic equation will be

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta \quad \text{where} \quad i^2 = -1.$$

$\beta > 0$

We want our solutions in the form of *real valued* functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x}.$$

We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number i.

Deriving the solutions Case III

Recall Euler's Formula[2] : $e^{i\theta} = \cos\theta + i\sin\theta$.

$$Y_1 = e^{\alpha x}e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$
$$= e^{\alpha x}\cos(\beta x) + i e^{\alpha x}\sin(\beta x)$$

$$Y_2 = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$
$$= e^{\alpha x}\cos(\beta x) - i e^{\alpha x}\sin(\beta x)$$

Let

$$y_1 = \frac{1}{2}(Y_1 + Y_2) = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$$

$$y_2 = \frac{1}{2i}(Y_1 - Y_2) = \frac{1}{2i}\left(2i\, e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$$

[2] As the sine is an odd function $e^{-i\theta} = \cos\theta - i\sin\theta$.

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

Let α be the real part of the complex roots and β be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example

Find the general solution of $$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0.$$

Char. eqn. $m^2 + 4m + 6 = 0$

Let's complete the square

$$m^2 + 4m + 4 - 4 + 6 = 0$$

$$(m+2)^2 + 2 = 0 \Rightarrow (m+2)^2 = -2$$

$$m+2 = \pm\sqrt{-2}$$

$$m+2 = \pm\sqrt{2}\, i$$

$$m = -2 \pm \sqrt{2}\, i$$

Complex case w/ $\alpha = -2$, $\beta = \sqrt{2}$

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t), \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

The general solution

$$x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$$

Higer Order Linear Constant Coefficient ODEs

$y = e^{mx}$

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.

- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.

- It may require a computer algebra system to find the roots for a high degree polynomial.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, m may be a root of multiplicity k where $1 \leq k \leq n$.

- If a real root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \dots, \quad x^{k-1} e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x} \cos(\beta x),\; e^{\alpha x} \sin(\beta x), \quad xe^{\alpha x} \cos(\beta x),\; xe^{\alpha x} \sin(\beta x), \dots,$$

$$x^{k-1} e^{\alpha x} \cos(\beta x),\; x^{k-1} e^{\alpha x} \sin(\beta x)$$

Example

Find the general solution of the ODE.

$y''' - 3y'' + 3y' - y = 0$

Ch. eqn: $m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)^3 = 0$

$m = 1$ triple root

$y_1 = e^{1x}$, $y_2 = x e^{1x}$, $y_3 = x^2 e^{1x}$

The general solution

$$\boxed{y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x}$$