February 25 Math 3260 sec. 51 Spring 2022

Section 2.2: Inverse of a Matrix

Question: Given an $n \times n$ matrix A, is there a matrix A^{-1} that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n.$$

If such matrix A^{-1} exists, we'll say that A is **nonsingular** (a.k.a. *invertible*). Otherwise, we'll say that A is **singular**.

Theorem

An $n \times n$ matrix A is invertible if and only if it is row equivalent to the identity matrix I_n. Moreover, if

$$\text{rref}(A) = E_k \cdots E_2 E_1 A = I_n, \quad \text{then} \quad A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[(E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces A to I_n, transforms I_n into A^{-1}.

This last observation—operations that take A to I_n also take I_n to A^{-1}—gives us a method for computing an inverse!

Algorithm for finding A^{-1}

To find the inverse of a given matrix A:

- Form the $n \times 2n$ augmented matrix $[A \quad I]$.
- Perform whatever row operations are needed to get the first n columns (the A part) to rref.
- If rref(A) is I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$, and the inverse A^{-1} will be the last n columns of the reduced matrix.
- If rref(A) is NOT I, then A is not invertible.

Remarks: We don't need to know ahead of time if A is invertible to use this algorithm.
If A is singular, we can stop as soon as it's clear that rref(A) $\neq I$.

Examples: Find the Inverse if Possible

(b) $A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$

Set up $[A \ I]$

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}\right]$$

$4R_1 + R_2 \to R_2$

$2R_1 + R_3 \to R_3$

$$\begin{array}{cccccc} 4 & 8 & -4 & 4 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{cccccc} 2 & 4 & -2 & 2 & 0 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cccccc} 0 & 2 & -2 & 8 & 2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}$$

$2R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 10 & 2 & 1 \end{array}\right]$$

It's not possible to get a nonzero entry in the third row third column. So A is not row equivalent to I, and hence A is not invertible.

Try it

Try to find A^{-1} where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by doing row reduction on the augmented matrix

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Section 2.3: Characterization of Invertible Matrices

Given an $n \times n$ matrix A, we can think of

- A matrix equation $A\mathbf{x} = \mathbf{b}$;
- A linear system that has A as its coefficient matrix;
- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$;
- Not to mention things like its **pivots**, its **rref**, the linear dependence/independence of its columns, blah blah blah...

Question: How is this stuff related, and how does being singular or invertible tie in?

Theorem: Suppose A is $n \times n$. The following are equivalent. [1]

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.

(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(j) There exists an $n \times n$ matrix C such that $CA = I$.

(k) There exists an $n \times n$ matrix D such that $AD = I$.

(l) A^T is invertible.

[1] Meaning all are true or none are true.

Theorem: (An inverse matrix is unique.)

Let A and B be $n \times n$ matrices. If $AB = I$, then A and B are both invertible with $A^{-1} = B$ and $B^{-1} = A$.

To show this, suppose $AB = I$ and consider the homogeneous equation

$$B\vec{x} = \vec{0}.$$

Multiply on the left by A.

$$AB\vec{x} = A\vec{0}$$

$$I\vec{x} = \vec{0} \implies \vec{x} = \vec{0}$$

$B\vec{x}=\vec{0}$ has only the trivial solution

hence B is invertible. ((d) $\Rightarrow$ (a))

B^{-1} exists. From $AB=I$, multiply

on the right by B^{-1}

$$ABB^{-1} = IB^{-1}$$

$$AI = IB^{-1} \implies A = B^{-1}.$$

Since B^{-1} is invertible A is invertible

and

$$A^{-1} = (B^{-1})^{-1} = B.$$