February 27 Math 2306 sec. 51 Spring 2023

Section 8: Homogeneous Equations with Constant Coefficients

We were considering linear, constant coefficient, homogeneous equations.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0$$

where each a_i is constant.

We assume that $y = e^{mx}$ for constant m, and obtain such solutions when m is a solution to the polynomial **characteristic equation**

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0$

- We have to get n linearly independent solutions.
- Complex roots have to appear in conjugate pairs.
- If m_i is a root of multiplicity k, we get k solutions

$$e^{m_i x}, \quad xe^{m_i x}, \quad x^2 e^{m_i x}, \ldots, x^{k-1} e^{m_i x}$$

- If $\alpha \pm i\beta$ is a complex conjugate pair of multiplicity k, we get $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$
$$x^{k-1} e^{\alpha x}\cos(\beta x),\ x^{k-1} e^{\alpha x}\sin(\beta x)$$

- The general solution is the linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

Example

Find the general solution of the ODE.

$y^{(4)} + 3y'' - 4y = 0$

4th order, a fundamental solution set will have four solutions.

Characteristic eqn

$$m^4 + 3m^2 - 4 = 0$$

factors

$$(m^2 + 4)(m^2 - 1) = 0$$

$$(m^2+4)(m-1)(m+1) = 0$$

$m^2+4=0 \implies m^2=-4$

$m = \pm\sqrt{-4}$

$m = \pm 2i$

complex $\alpha \pm i\beta$

$\alpha = 0, \quad \beta = 2$

$y_1 = e^{0x}\cos(2x) = \cos(2x)$

$y_2 = e^{0x}\sin(2x) = \sin(2x)$

$m-1=0 \implies m=1$

$y_3 = e^{1x}$.

$m+1=0 \implies m=-1$

$$y_4 = e^{-1x}$$

The general solution is

$$y = C_1 \cos(2x) + C_2 \sin(2x) + C_3 e^{x} + C_4 e^{-x}$$

Example

Consider the ODE

$$y^{(6)} - 6y^{(4)} + 104y''' + 9y'' - 312y' + 2704y = 0.$$

The characteristic equation is

$$m^6 - 6m^4 + 104m^3 + 9m^2 - 312m + 2704 = 0$$

which factors as

$$((m-2)^2 + 9)^2 (m+4)^2 = 0.$$

Find the general solution.

A fundamental solution set
will have six, linearly independent
solutions

$$((m-2)^2+9)^2(m+4)^2 = 0$$

$(m+4)^2 = 0 \Rightarrow m+4=0 \Rightarrow m=-4$ double root

$y_1 = e^{-4x}$, $y_2 = xe^{-4x}$

$((m-2)^2+9)^2 = 0 \Rightarrow (m-2)^2+9=0$

$(m-2)^2 = -9$

$m-2 = \pm\sqrt{-9}$

$m-2 = \pm 3i$

$m = 2 \pm 3i$ complex pair

both double roots

We get 4 solutions

$$y_3 = e^{2x}\cos(3x) \qquad y_5 = x e^{2x}\cos(3x)$$

$$y_4 = e^{2x}\sin(3x) \qquad y_6 = x e^{2x}\sin(3x)$$

The general solution is

$$y = C_1 e^{-4x} + C_2 x e^{-4x} + C_3 e^{2x}\cos(3x) + C_4 e^{2x}\sin(3x) +$$

$$+ C_5 x e^{2x}\cos(3x) + C_6 x e^{2x}\sin(3x).$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e^{kx}
- sines and/or cosines, $\sin(kx)$ or $\cos(kx)$
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example[1]

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

The left is constant coef. and the right is a polynomial.

$g(x) = 8x+1$ 1st degree polynomial

We'll focus on y_p. Let's "guess" that y_p is the same kind of function as $g(x)$. i.e. y_p is a 1st degree poly.

[1] **We're only ignoring the y_c part to illustrate the process.**

$$y'' - 4y' + 4y = 8x + 1$$

Suppose $y_p = Ax + B$. Sub this in.

$y_p' = A$

$y_p'' = 0$

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$0 - 4(A) + 4(Ax + B) = 8x + 1$$

$$4Ax + (-4A + 4B) = 8x + 1$$

Match like terms

$$4A = 8$$

$$-4A + 4B = 1$$

$A=2$, $\quad 4B = 1+4A = 1+4(2) = 9$

$$B = \frac{9}{4}$$

So $\quad y_p = 2x + \frac{9}{4}$ is a

particular solution.

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We found the particular solution

$$y_p = 2x + \frac{9}{4}$$

by

- guessing that y_p is the same kind of function as g,
- setting it up with **undetermined** coefficients (A, B, etc.), and
- substituting it into the ODE to find the coefficients that work.

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

Focus on y_p only.

$g(x) = 6e^{-3x}$ constant times e^{-3x}

Set $y_p = Ae^{-3x}$ sub into ODE

$y_p' = -3Ae^{-3x}$

$y_p'' = 9Ae^{-3x}$

$$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$$

$$9Ae^{-3x} - 4\left(-3Ae^{-3x}\right) + 4\left(Ae^{-3x}\right) = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

matching like terms

$$25A = 6$$

$$A = \frac{6}{25}$$

$$\boxed{\text{So} \quad y_p = \frac{6}{25}e^{-3x}}$$

The Initial Guess Must Be General in Form

Find a particular solution to $\quad y'' - 4y' + 4y = 16x^2$

$g(x) = 16x^2$ monomial w/ x^2

quadratic

Suppose we consider g to be a monomial.

Set $y_p = Ax^2$ sub

$y_p' = 2Ax$

$y_p'' = 2A$

$y_p'' - 4y_p' + 4y_p = 16x^2$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$\underline{4A}\,x^2 - \underline{8A}\,x + \underline{2A} = \underline{16}x^2 + 0x + 0$$

matching

$$\left.\begin{aligned} 4A &= 16 \\ -8A &= 0 \\ 2A &= 0 \end{aligned}\right\} \Rightarrow \quad \overset{\text{false}}{4 = 0}$$

Our initial guess was wrong! We didn't account for x or constant terms.

$$y_p = Ax^2 + Bx + C$$

This would be the correct form for the particular solution.