February 28 Math 3260 sec. 51 Spring 2024

Section 3.2: Properties of Determinants

Theorem:

Let A be an $n \times n$ matrix, and suppose the matrix B is obtained from A by performing a single elementary row operation[a]. Then

(i) If B is obtained by adding a multiple of a row of A to another row of A (row replacement), then

$$\det(B) = \det(A).$$

(ii) If B is obtained from A by swapping any pair of rows (row swap) , then

$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by scaling any row by the constant k (scaling), then

$$\det(B) = k\det(A).$$

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[a]If "row" is replaced by "column" in any of the operations, the conclusions

Example

Let $\quad A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 6 \\ 1 & -3 & 1 & 4 \\ 1 & 5 & 1 & -5 \end{bmatrix}$

The following row operations produce the ref matrix B.

1. $-R_1 + R_2 \to R_2$ no change
2. $-R_1 + R_3 \to R_3$ "
3. $-R_1 + R_4 \to R_4$ "
4. $R_2 \leftrightarrow R_4$ factor -1
5. $R_2 + R_3 \to R_3$ no change
6. $R_3 \leftrightarrow R_4$ factor -1
7. $-2R_3 + R_4 \to R_4$ no change
8. $\frac{1}{13}R_4 \to R_4$ factor of $\frac{1}{13}$

Example

$$\text{ref}(A) = B = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 4 & -1 & -8 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\det(B) = 1(4)(-1)(1) = -4$

B is upper triangular.

Find $\det(A)$ by relating it to $\det(B)$.

$$\det(B) = (-1)(-1)\left(\frac{1}{13}\right)\det(A)$$

$$-4 = \frac{1}{13}\det(A)$$

$$\Longrightarrow \quad \det(A) = -4(13) = -52$$

Results on Determinants

Theorem

The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem

For $n \times n$ matrix A, $\det(A^T) = \det(A)$.

Theorem

For $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

Example

Show that if A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof: Suppose A is invertible.

Then $A^{-1}A = I$

$$\det(A^{-1}A) = \det(I) = 1$$

by the last theorem $\quad \det(A^{-1})\det(A) = 1$

$$\Rightarrow \quad \det(A^{-1}) = \frac{1}{\det(A)}.$$