February 28 Math 3260 sec. 52 Spring 2022

Section 2.3: Characterization of Invertible Matrices

Theorem: Suppose A is $n \times n$. The following are equivalent.

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.

(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(j) There exists an $n \times n$ matrix C such that $CA = I$.

(k) There exists an $n \times n$ matrix D such that $AD = I$.

(l) A^T is invertible.

Example

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a one to one linear transformation. Can we determine whether T is onto? Why (or why not)?

The standard matrix A for T would be $n \times n$. So $\vec{x} \mapsto A\vec{x}$ is one to one.

By the previous theorem, (f) $\Rightarrow$ (i),

$\vec{x} \mapsto A\vec{x}$ is also onto. So we can say that T is onto.

Invertible Linear Transformations

Definition: A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that both

$$(S\circ T)(\vec{x}) = S(T(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad T(S(\mathbf{x})) = \mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

If such a function exists, we typically denote it by

$$S = T^{-1}.$$

Theorem (Invertibility of a linear transformation and its matrix)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and A its standard matrix. Then T is invertible if and only if A is invertible. Moreover, if T is invertible, then

$$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

Example

Use the standard matrix to determine if the linear transformation is invertible. If it is invertible, characterize the inverse transformation.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \text{given by} \quad T(x_1, x_2) = (3x_1 - x_2, 4x_2).$$

The standard matrix A will be 2x2.

$$A = \left[T(\vec{e}_1) \;\; T(\vec{e}_2) \right]$$

$$T(\vec{e}_1) = T(1,0) = (3\cdot 1 - 0, 4\cdot 0) = (3,0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = T(0,1) = (3\cdot 0 - 1, 4\cdot 1) = (-1,4) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}. \quad \det(A) = 3(4) - 0(-1) = 12 \neq 0$$

So A is invertible making T invertible

and $T^{-1}(\vec{x}) = A^{-1}\vec{x}$.

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$$

$$T^{-1}(\vec{x}) = A^{-1}\vec{x} = \frac{1}{12} \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 4x_1 + x_2 \\ 3x_2 \end{bmatrix}$$

In the original format

$$T^{-1}(x_1, x_2) = \left(\frac{1}{3}x_1 + \frac{1}{12}x_2 \, , \, \frac{1}{4}x_2 \right)$$

$$T(x_1, x_2) = (3x_1 - x_2,\ 4x_2)$$

$$T^{-1}(x_1, x_2) = \left(\tfrac{1}{3}x_1 + \tfrac{1}{12}x_2,\ \tfrac{1}{4}x_2\right)$$

$$(T^{-1} \circ T)(\vec{x}) = T^{-1}(T(\vec{x}))$$

$$= T^{-1}(T(x_1, x_2))$$

$$= T^{-1}(3x_1 - x_2,\ 4x_2)$$

$$= \left(\tfrac{1}{3}(3x_1 - x_2) + \tfrac{1}{12}(4x_2),\ \tfrac{1}{4}(4x_2)\right)$$

$$= \left(x_1 - \tfrac{1}{3}x_2 + \tfrac{1}{3}x_2,\ x_2\right)$$

$$= (x_1, x_2)$$