February 2 Math 3260 sec. 52 Spring 2022

Section 1.5: Solution Sets of Linear Systems

We defined a linear system as being **homogeneous** if the right hand side is zero (the zero vector). We have theorems on homogeneous systems that say:

Theorem: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, $\mathbf{x} = \mathbf{0}$, called the **trivial solution.**

Theorem: The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Examples

Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

$$\text{(b)}\quad \begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

We set up and row reduced the augmented matrix to get

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and concluded that $x_1 = \frac{4}{3}x_3$, $x_2 = 0$, and x_3 is free.

Examples

Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(b)
$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

We can express the solution in the form

$$\mathbf{x} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \quad \text{where } x_3 \text{ is any real number.}$$

(c) $x_1 - 2x_2 + 5x_3 = 0$

The augmented matrix is

$\begin{bmatrix} 1 & -2 & 5 & 0 \end{bmatrix}$ This is an rref

non pivot columns

x_1 is basic, x_2 and x_3 are free

There are nontrivial solutions. The solutions are given by

$x_1 = 2x_2 - 5x_3$

x_2 and x_3 are free

The solution vectors $\vec{x}$ have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

when x_2 and x_3 are any real numbers

we can say $\vec{x}$ is in

$$\text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form $\mathbf{x} = x_3\mathbf{v}$. Example (c)'s solution set consisted of vector that look like $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$. Since these are **linear combinations**, we could write the solution sets like

$$\text{Span}\{\mathbf{v}\} \quad \text{or} \quad \text{Span}\{\mathbf{u}, \mathbf{v}\}.$$

Instead of using the variables x_2 and/or x_3 we often substitute **parameters** such as s or t.
The forms

$$\mathbf{x} = s\mathbf{u}, \quad \text{or} \quad \mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

are called **parametric vector forms.**

Geometry

The **parametric vector form** of the solution set of the system

$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array} \text{ is}$$

$$\mathbf{x} = s\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

↑ s is an element of the real numbers

This is a line in $\mathbb{R}^3$ through the points $(0,0,0)$ and $\left(\frac{4}{3},0,1\right)$.

Geometry

Span{$\vec{u}$} is the set of all vectors of the form $c\vec{u}$ where c is any real number.

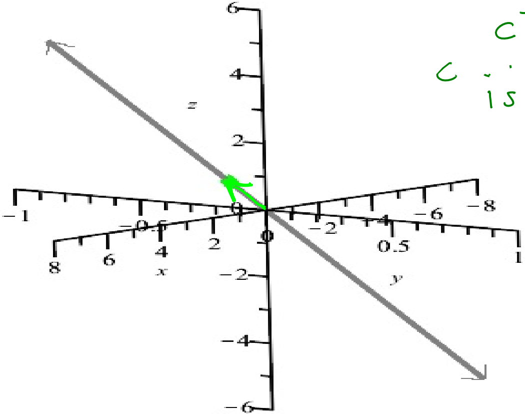

Figure: Plot of the line $\mathbf{x} = s\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$. The point $\left(\frac{4}{3}, 0, 1\right)$ is shown in green.

Geometry

The **parametric vector form** of the solution set of
$x_1 \; - \; 2x_2 \; + \; 5x_3 \; = \; 0$ is

$$\mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

This is a plane in $\mathbb{R}^3$ that contains the points $(0,0,0)$, $(2,1,0)$, and $(-5,0,1)$.

Geometry

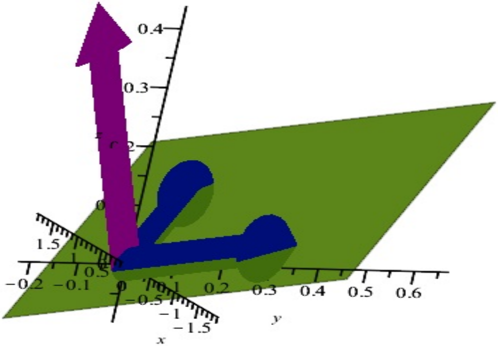

Figure: Plot of the plane $\mathbf{x} = s\begin{bmatrix}2\\1\\0\end{bmatrix} + t\begin{bmatrix}-5\\0\\1\end{bmatrix}$. The blue vectors are in the directions of $(2, 1, 0$ and $(-5, 0, 1)$. The purple vector is perpendicular to the plane.