Feb. 3 Math 2254H sec 015H Spring 2015

Section 7.4: Rational Functions, Partial Fractions

Theorem: Every polynomial $Q(x)$ with real coefficients can be factored into a product

$$Q(x) = q_1(x)q_2(x)\cdots q_k(x)$$

where each q_i is either a linear factor (i.e. $q_i(x) = ax + b$) or an irreducible quadratic (i.e. $q_i(x) = ax^2 + bx + c$ where $b^2 - 4ac < 0$).

Decomposing Proper Rational Functions

Let $f(x) = P(x)/Q(x)$ be a **proper** rational function, and let $Q(x)$ be factored completely into linear and irreducible quadratic factors

$$f(x) = \frac{P(x)}{q_1(x)q_2(x)\cdots q_k(x)}.$$

We'll consider four cases

(i) each factor of Q is linear and none are repeated,

(ii) each factor of Q is linear and one or more is repeated,

(iii) some factor(s) of Q are quadratic, but no quadratic is repeated,

(iv) Q has at least one repeated quadratic factor.

Case (i) Non-repeated Linear Factors

Suppose $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$. And no pair of a's and b's (both) match. Then we look for a decomposition of f in the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

For example

$$\frac{10 - x}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4}.$$

Note that each fraction in the expansion is a proper rational function with denominator a line.

Case (ii) A Repeated Linear Factor

Suppose $Q(x)$ has only linear factors, but that one of them is repeated. That is, suppose $(a_i x + b_i)^n$ is a factor of Q. Then for this term, the decomposition of f will contain the n terms

$$\frac{A_{i1}}{a_i x + b_i} + \frac{A_{i2}}{(a_i x + b_i)^2} + \cdots + \frac{A_{in}}{(a_i x + b_i)^n}.$$

For example,

$$\frac{3x^2 + 2x - 1}{(x+1)^2(x-2)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{E}{(x-2)^3}.$$

Example: Evaluate the integral

$$\int \frac{7x^2+7x+4}{x(x+1)^2}\,dx$$

Partial frac. Decomp

$$\frac{7x^2+7x+4}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$7x^2+7x+4 = A(x+1)^2 + Bx(x+1) + Cx$$

$$7x^2+7x+4 = A(x^2+2x+1) + B(x^2+x) + Cx$$

$$= (A+B)x^2 + (2A+B+C)x + A$$

$$A+B = 7 \Rightarrow B = 7-4 = 3$$

$$2A+B+C = 7 \qquad C = 7-3-2\cdot 4 = -4$$

$$A = 4$$

$$\boxed{\begin{aligned} A&=4\\ B&=3\\ C&=-4 \end{aligned}}$$

$$\int \frac{7x^2+7x+4}{x(x+1)^2}\,dx = \int \left(\frac{4}{x} + \frac{3}{x+1} - \frac{4}{(x+1)^2} \right) dx$$

If $u = x+1$

$du = dx$

$$= 4\ln|x| + 3\ln|x+1| - 4\int u^{-2}\,du$$

$$= 4\ln|x| + 3\ln|x+1| - 4\,\frac{u^{-1}}{-1} + C$$

$$= 4\ln|x| + 3\ln|x+1| + \frac{4}{x+1} + C$$

Case (iii) Nonrepeated Quadratic Factors

Suppose $Q(x)$ has a factor of the form $q(x) = ax^2 + bx + c$ with $(b^2 - 4ac < 0)$ that is not repeated (appears only to the first power). Then for this term, the decomposition of f will contain the term

$$\frac{Ax + B}{ax^2 + bx + c}.$$

For example

$$\frac{3x+7}{(x+1)(x-2)(x^2+4)(x^2+x+1)} =$$

$$= \frac{A}{x+1} + \frac{B}{x-2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{x^2+x+1}.$$

Note that the most **general proper** rational function with a quadratic denominator will have a line in the numerator! It may be that one of A or B is zero, but we don't assume any such thing up front!

Example: Evaluate the Integral

$$\int \frac{3x+4}{(x-1)(x^2+1)}\,dx$$

Partial Fraction Decomp

$$\frac{3x+4}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$3x+4 = A(x^2+1) + (Bx+C)(x-1)$$

$$3x+4 = (A+B)x^2 + (-B+C)x + A-C$$

$A+B = 0 \Rightarrow A = -B$

$-B+C = 3$

$A - C = 4$

$-B + C = 3$

$\underline{-B - C = 4}$

$-2B = 7$

$B = \frac{-7}{2}$, $A = \frac{7}{2}$ $C = 3 + \left(\frac{-7}{2}\right) = \frac{-1}{2}$

$$\int \frac{3x+4}{(x-1)(x^2+1)}\,dx = \int \left(\frac{7/2}{x-1} + \frac{-\frac{7}{2}x - \frac{1}{2}}{x^2+1} \right) dx$$

$$= \frac{7}{2}\int \frac{dx}{x-1} - \frac{7}{2}\int \frac{x}{x^2+1}\,dx - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{7}{2}\ln|x-1| - \frac{7}{4}\int \frac{du}{u} - \frac{1}{2}\tan^{-1}x + C$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{7}{2}\ln|x-1| - \frac{7}{4}\ln|u| - \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{7}{2}\ln|x-1| - \frac{7}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

Case (iv) Repeated Quadratic Factor

Suppose $Q(x)$ has a factor of the form $q(x) = (ax^2 + bx + c)^r$ with $(b^2 - 4ac < 0)$ with r an integer bigger than 1. Then for this term, the decomposition of f will contain the r terms

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}.$$

For example

$$\frac{1}{(x+1)^2(x^2+4)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx + D}{x^2+4} + \frac{Ex + F}{(x^2+4)^2} + \frac{Gx + H}{(x^2+4)^3}$$

Example: Evaluate the Integral

$$\int \frac{3x^2 - x + 12}{(x^2+4)^2}\,dx$$

Partial Fraction Decomp

$$\frac{3x^2 - x + 12}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$$3x^2 - x + 12 = (Ax+B)(x^2+4) + Cx + D$$

$$= Ax^3 + Bx^2 + (4A+C)x + 4B + D$$

$A = 0$ $\quad$ $4A + C = -1 \Rightarrow C = -1$

$B = 3$ $\quad$ $4B + D = 12 \Rightarrow D = 12 - 3\cdot 4 = 0$

$$\int \frac{3x^2 - x + 12}{(x^2+4)^2}\,dx = \int \left(\frac{3}{x^2+4} - \frac{x}{(x^2+4)^2} \right) dx$$

$u = x^2 + 4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\int u^{-2}\,du$$

$$= \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\,\frac{u^{-1}}{-1} + C$$

$$= \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2(x^2+4)} + C$$

$$\frac{3x^2 - x + 12}{(x^2+4)^2} = \frac{3(x^2+4) - x}{(x^2+4)^2}$$

$$= \frac{3}{x^2+4} - \frac{x}{(x^2+4)^2}$$

Improper Rational Functions

Evaluate the integral $$\int_0^1 \frac{4x^4 + x^3 + 3x^2 + 4x + 2}{x^3 + x^2 + x + 1}\, dx$$

$$\begin{array}{r} 4x - 3 \\ x^3+x^2+x+1 \,\overline{\big)\, 4x^4+x^3+3x^2+4x+2} \\ -(4x^4+4x^3+4x^2+4x) \\ \hline -3x^3 - x^2 \qquad + 2 \\ -(-3x^3-3x^2-3x-3) \\ \hline 2x^2+3x+5 \end{array}$$

$$\frac{P(x)}{Q(x)} = 4x-3 + \frac{2x^2+3x+5}{x^3+x^2+x+1} = 4x-3 + \frac{2x^2+3x+5}{(x+1)(x^2+1)}$$

$$x^3 + x^2 + x + 1 = (x+1)(x^2+1)$$

Partial Frac. Decomp:

$$\frac{2x^2+3x+5}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$2x^2 + 3x + 5 = A(x^2+1) + (Bx+C)(x+1)$$

$$= (A+B)x^2 + (B+C)x + A + C$$

$$\left.\begin{aligned} A + B \quad &= 2 \\ B + C &= 3 \\ A \quad + C &= 5 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} A - C &= -1 \\ A + C &= 5 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 2 \\ C &= 3 \\ B &= 0 \end{aligned}$$

$$\int_0^1 \frac{4x^4+x^3+3x^2+4x+2}{(x+1)(x^2+1)}\,dx = \int_0^1 \left(\frac{2}{x+1} + \frac{3}{x^2+1}\right)dx + \int_0^1 (4x-3)\,dx$$

$$= 2\ln|x+1| + 3\tan^{-1}(x)\Big|_0^1 + \left[2x^2 - 3x\right|_0^1$$

$$= 2\ln 2 - 2\ln 1 + 3\tan^{-1}(1) - 3\tan^{-1}(0) + (-1-0)$$

$$= 2\ln 2 + \frac{3\pi}{4} - 1$$