February 5 Math 3260 sec. 51 Spring 2024

Section 1.8: Intro to Linear Transformations

Recall that the product $A\mathbf{x}$ is a vector that is a linear combination of the columns of A.

If the columns of A are vectors in $\mathbb{R}^m$, and there are n of them, then

- A is an $m \times n$ matrix,
- the product $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbb{R}^n$, and
- the vector $\mathbf{b} = A\mathbf{x}$ is a vector in $\mathbb{R}^m$.

Remark: We can think of a matrix A as an **operator that acts** on vectors $\mathbf{x}$ in $\mathbb{R}^n$ (via the product $A\mathbf{x}$) to produce vectors $\mathbf{b}$ in $\mathbb{R}^m$.

Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Remark

Another name for a *transformation* is a **function** or **mapping**. The parentheses notation $T(\cdot)$ is typical function notation. A transformation takes a vector as an input and spits out a vector as the output.

Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Function Notation: If a transformation T takes a vector $\mathbf{x}$ in $\mathbb{R}^n$ and maps it to a vector $T(\mathbf{x})$ in $\mathbb{R}^m$, we can write

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

which reads "T maps $\mathbb{R}^n$ into $\mathbb{R}^m$."

And we can write

$$\mathbf{x} \mapsto T(\mathbf{x})$$

which reads "**x** maps to T of **x**."

The following vertically stacked notation is often used:

$$\begin{aligned} T : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \mathbf{x} &\mapsto T(\mathbf{x}) \end{aligned}$$

Key Terms

For $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

- $\mathbb{R}^n$ is the **domain**, and
- $\mathbb{R}^m$ is called the **codomain.**
- For $\mathbf{x}$ in the domain, $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ under T. (We can call $\mathbf{x}$ a **pre-image** of $T(\mathbf{x})$.)
- The collection of all images is called the **range.**
- If $T(\mathbf{x})$ is defined by multiplication by the $m \times n$ matrix A, we may denote this by $\mathbf{x} \mapsto A\mathbf{x}$.

Matrix Transformation Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$. Define the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by the mapping $T(\mathbf{x}) = A\mathbf{x}$.

(a) Find the image of the vector $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ under T.

Find $T(\vec{u})$.

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ 6 \end{bmatrix}$$

Example Continued...

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}, \qquad \begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ x &\mapsto A\mathbf{x} \end{aligned}$$

(b) Determine a vector $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$.

This is asking us to find $\vec{x}$ in $\mathbb{R}^2$ such that $T(\vec{x}) = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$. This gives a matrix equation

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$$

Using an augmented matrix

$$\begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 2 \\ x_2 = -2 \end{array}$$

Hence $\vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is a preimage of $\begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$

$$T\left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}.$$

Example Continued...

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}, \qquad \begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ x &\mapsto A\mathbf{x} \end{aligned}$$

(c) Determine if $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in the range of T.

This is asking whether there exists $\vec{x}$ in $\mathbb{R}^2$ such that $T(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Using an augmented matrix

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The system $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is inconsistent.

Hence $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not in the range of T.

Linear Transformations

Definition

A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, is **linear** provided

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

Remark 1: These were the two properties (that I claimed were a *big deal*) of the product $A\mathbf{x}$ from section 1.4.

Remark 2: Every matrix transformation (e.g. $\mathbf{x} \mapsto A\mathbf{x}$) is a linear transformation. And every linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be stated in terms of a matrix.